CONTINUED RECIPROCAL ROOTS

DIXON J. JONES

ABSTRACT. For p a real number, a continued pth power is an expression of the form $\lim_{n\to\infty} a_0 + (a_1 + (a_2 + (\ldots + (a_n)^p \ldots)^p)^p)^p$. Continued reciprocal roots are the special case in which $-1 < p < 0$. Following a brief historical survey, we derive a sharp necessary and sufficient divergence condition for continued reciprocal qth roots of positive terms, where $q = |1/p|$.

1. INTRODUCTION

A curious hybrid of the continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} \tag{1}$$

and the continued square root, or continued radical,

$$a_0 + \sqrt{a_1 + \sqrt{a_2 + \sqrt{\cdots}}} \tag{2}$$

is the *continued reciprocal square root*

$$a_0 + \cfrac{1}{\sqrt{a_1 + \cfrac{1}{\sqrt{a_2 + \cfrac{1}{\sqrt{\ddots}}}}}} \,. \tag{3}$$

This leggy expression is easily spotted on the page, but difficult to find in the literature. Indeed, the evidence to date suggests that continued reciprocal roots have appeared only a handful of times over the past 200 years. In this paper we survey the few known works that touch on expression (3), review some of its basic properties, and prove the following divergence condition.

Theorem 1. *For $0 < p < 1$ and $q = 1/p$, the continued reciprocal qth root with terms $a_i > 0$, $i = 0, 1, 2, \ldots$, diverges if, and only if,*

$$\limsup_{i\to\infty} a_i^{p^i} < 1 \,.$$

Roughly speaking, this says that a continued reciprocal root diverges when its terms decrease too rapidly. As we will see, Theorem 1 is analogous in form to a sharp necessary and sufficient *convergence* condition for continued radicals of nonnegative terms.

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2. HISTORY

Continued fractions arose in antiquity out of arithmetical problems [4], and nested square root expressions first appeared in the late 16th century in connection with polygonal approximations of a circle [33]. But continued radicals and continued reciprocal roots can reasonably be said to have originated in the early 1800s, with the inception of "successive substitution" as a method for "solving" equations (see for instance [24, 30]). The idea was that a solution to the equation $x = t(x)$ should emerge by repeatedly substituting $t(x)$ for the argument of t:

$$x = t(t(x)) \ ,$$
$$x = t(t(t(x))) \ ,$$

and so on. Under the right conditions, this amounts to the computation of a fixed point of t by forward iteration.

Among those intrigued with this process was Christian Doppler—the very Doppler whose name is now synonymous with wave propagation effects and radar devices. In a paper published in 1832 [9], Doppler proposed the exceedingly general continued radical

$$A\sqrt[a]{\ }[\alpha + B\sqrt[b]{\ }(\beta + C\sqrt[c]{\ }(\gamma + D\sqrt[d]{\ }(\delta + E\sqrt[e]{\ }(\varepsilon + \ldots \text{in infinitum}))))] \ , \qquad (4)$$

and noted that "für eine gewisse Annahme der Exponenten [for a certain assumption of exponents]" one obtains the continued fraction

$$\frac{A}{\alpha + \frac{B}{\beta + \frac{C}{\gamma + \frac{D}{\delta + \text{etc.}}}}} \ ,$$

and also

$$\frac{A}{\sqrt[a]{\alpha + \frac{B}{\sqrt[b]{\beta + \frac{C}{\sqrt[c]{\gamma + \frac{D}{\sqrt[d]{\delta + \text{etc.}}}}}}}}} \ . \qquad (5)$$

Though Doppler did not identify it by name, (5) may well be the first continued reciprocal root to appear in print. Doppler specified that the independent variables—the terms, multipliers, and exponents, along with their signs—be finite in number, and that they occur in a periodic arrangement. This made the continued radical a product of successive substitution: If the expression's period was k, then one essentially constructed an iterated solution to the equation $x = t_1(t_2(\cdots(t_k(x))\cdots)) = T(x)$. However, Doppler then turned his attention exclusively to continued square roots, and made no further reference to his visionary special cases.

Throughout the 19th and early 20th centuries, the method of successive substitution received much attention (with varying degrees of rigor) in the search for solutions to general trinomial and other equations [7, 8, 10, 24]. But after Doppler and before 1900, continued reciprocal roots apparently turned up just once. The expression

$$x = \sqrt[m]{b + \cfrac{}{a + \sqrt[m/n]{b + \cfrac{}{a + \sqrt[m/n]{b + \cfrac{}{a + \sqrt[m/n]{\cdots}}}}}}} \; . \tag{6}$$

appeared as a "solution" to the trinomial equation $x^{m+n} + ax^m = b$ in Günther's 1880 note [12]. Hoffmann [14] and Isenkrahe [15] followed in 1881 and 1888, respectively, with more thorough examinations of Günther's algorithms. These three papers mark the last public appearance of continued reciprocal roots for more than a hundred years.

On the other hand, continued radicals were (and are) of perennial interest.[1] Still tied to successive substitution until about 1910, they dealt exclusively with constant terms in periodic arrangements, and often appeared in geometric and trigonometric contexts [2, 5, 6, 22, 34]. Although his proof was flawed, Ramanujan's identity

$$4 = \sqrt{6 + 2\sqrt{7 + 3\sqrt{8 + \cdots}}}$$

from 1911 [27] is noteworthy as the first continued radical with non-periodic terms. The first rigorous treatment of continued radicals of arbitrary, nonnegative terms was given in a 1916 journal problem by Pólya [25] (solved the following year by Szegö, and included in their famous 1925 problem book [26]). Pólya and Szegö showed that a continued square root with positive real terms a_i converges or diverges accordingly as

$$\limsup_{i\to\infty} \frac{\log\log a_i}{i}$$

is less than or greater than 2. This was sharpened in 1935 by Herschfeld, who presented a general convergence theorem [13, Theorem III] that includes the following as a special case.

Theorem 2 (Herschfeld 1935)**.** *For $0 < p < 1$ and $q = 1/p$, the continued qth root with terms $a_i \geq 0$, $i = 0, 1, 2, \ldots$, converges if, and only if,*

$$\limsup_{i\to\infty} a_i^{p^i} < +\infty \, .$$

Herschfeld's result (the model upon which our Theorem 1 is based) was independently rediscovered at least twice: by Sizer in 1986 [31], and in 1990 by the late German analyst Detlef Laugwitz [20], with whom we rejoin the trail of continued reciprocal roots.[2] Laugwitz investigated what he called *Kettenoperationen* or *chain operations*, that is, the sequence $(P_n)_{n=1}^{\infty}$ defined by

$$P_n = f(a_1 + f(a_2 + f(a_3 + f(\ldots + f(a_n))))) \, , \tag{7}$$

where f maps the positive reals to themselves and the a_i are arbitrary positive real numbers.[3] Laugwitz used the example of continued radicals to develop convergence conditions

[1]Even in 1877, Realis [28] could dismiss a long paper from 1862 [3] with the remark "[L]a théorie des radicaux continus est loin d'être nouvelle, et M. A. Bouché n'est pas le premier qui s'en soit occupé. [The theory of continued radicals is far from new, and Monsieur A. Bouché is not the first to have studied it.]"

[2]Indeed, Laugwitz was the first to suggest both the German "reziproke Kettenwurzel" and the English "continued reciprocal root."

[3]Such constructions had been proposed as early as 1924 [1, 11, 19, 32], although Laugwitz does not mention them.

for (7) when f is a continuous and monotonically increasing or decreasing function. In the latter case he proved

Proposition 1 (Laugwitz 1990)**.** *Let f be monotone decreasing on the positive reals, $f'(x) \to 0$ for $x \to \infty$, and for $0 < c < 1$ suppose there exists an $\overline{x} = \overline{x}(c)$ such that $|f'(x)| \leq c$ for $x \geq \overline{x}$. Then P_n converges if all $a_n \geq \overline{x}$ and*

$$|P_{n+1} - P_n| \leq c^n f(\overline{x}) \ .$$

From this he was able to show that a continued reciprocal square root converges if $\liminf a_n > \sqrt[3]{2}/2$; but he admitted that, for functions of the form $f(x) = x^{-p}$, Proposition 1 yields merely a "groben Abschätzung [rough estimate]." Laugwitz gave special attention to chain operations of constant terms, and proved that a continued reciprocal qth root of positive constant terms converges. He also showed that, for decreasing f, divergent chain operations exist. Laugwitz's student Schönefuss further explored these ideas in his 1992 thesis [29], and gave a convergence condition somewhat stronger than Proposition 1 for continued reciprocal qth roots. (We state the result as revised jointly by Laugwitz and Schönefuss in 1999 [21].)

Proposition 2 (Schönefuss 1992)**.** *For $0 < p < 1$ and $q = 1/p$, the continued reciprocal qth root with terms $a_i > 0$ converges if (but not only if)*

$$\lim_{i\to\infty} \frac{p^{i+1}}{a_i a_{i+1}^p} = 0 \ .$$

Continued reciprocal roots appeared again in a 2004 paper by Martin [23], who proved a number of exotic—but computationally impractical—properties of the f-expansion of a real number.[4] Martin's paper closes with several numerical examples, including continued reciprocal square root representations of some rational numbers. He shows, for instance, that the expansions of $\frac{2}{3}$ and $\frac{27}{47}$ terminate, whereas it is only conjectured, and likely difficult to prove, that the expansion for $\frac{3}{4}$ is nonterminating.

Our historical overview would be incomplete without an acknowledgment that, given the burgeoning store of archival material in digital formats, it is almost certain that more (and earlier) sources will come to light. Still, the readily accessible data points on the continued reciprocal roots timeline have thus far proven sparse indeed.[5]

3. Preliminaries

A *continued pth power* is an expression of the form

$$\lim_{n\to\infty} a_0 + (a_1 + (a_2 + (\ldots + (a_n)^p \ldots)^p)^p)^p \ . \tag{8}$$

Following Schönefuss [29], we use a standard continued fraction notation to write this as

$$\operatorname*{K}_{i=0}^{\infty} a_i^p = \lim_{i\to\infty} a_0 + (a_1 + (a_2 + (\ldots + (a_i)^p \ldots)^p)^p)^p \ .$$

[4]Anticipated some 20 years earlier by Kakeya [19], the f-expansion was introduced in 1944 by Bissinger [1] as a generalization of the continued fraction expansion. Though notationally identical to (7), the terms of an f-expansion are assumed to be integers.

[5]On the other hand, the references given here regarding continued radicals are a small sampling from an extensive literature. The online document [18] lists many other works on continued radicals from the 19th and early 20th centuries.

A continued pth power's nth *approximant* is

$$\operatorname*{K}_{i=0}^{n} a_i^p = a_0 + (a_1 + (a_2 + (\ldots + (a_n)^p \ldots)^p)^p)^p \,.$$

However, we usually write these expanded expressions with the exponents on the left:

$$\begin{aligned}\operatorname*{K}_{i=0}^{\infty} a_i^p &= \lim_{i\to\infty} a_0 + {}^p\!\big(a_1 + {}^p\!\big(a_2 + {}^p\!\big(\ldots + {}^p\!\big(a_i\big)\big)\big)\big) \\ &= a_0 + {}^p\!\big(a_1 + {}^p\!\big(a_2 + {}^p\!\big(\ldots\big)\big)\big) \,,\end{aligned}$$

to emphasize and streamline the characteristic right-to-left order of operations,[6] and we denote the pth power of a by both

$$a^p \quad \text{and} \quad {}^p\!(a) \,.$$

In what follows, we are mainly concerned with continued $-p$th powers in which $0 < p < 1$ (that is, continued reciprocal qth roots, where $q = 1/p$), and whose terms are positive real numbers.

For $m \geq 0$ the expression

$$\operatorname*{K}_{i=m}^{n} a_i^p = a_m + {}^p\!\big(a_{m+1} + {}^p\!\big(a_{m+2} + {}^p\!\big(\ldots + {}^p\!\big(a_n\big)\big)\big)\big)$$

is the (m,n)th *tail* (or, more loosely, a *finite tail*) of the continued power; the nth approximant is therefore the $(0,n)$th tail. When p and the a_i are understood, we abbreviate continued powers and their approximants and tails as K_0^∞, K_0^n, and K_m^n, respectively. The sequence of tails

$$\left(\operatorname*{K}_{m}^{m}, \operatorname*{K}_{m}^{m+1}, \operatorname*{K}_{m}^{m+2}, \ldots\right)$$

exhibits a characteristic behavior in its alternating elements when p is negative. The following result, well-known for continued fractions, holds for any continued $-p$th power.

Proposition 3. *Given $p > 0$ and integers m and n with $0 \leq m < n$, the finite tails of the continued $-p$th power with terms $a_i > 0$ satisfy*

$$0 < \operatorname*{K}_{m}^{m} < \operatorname*{K}_{m}^{m+2} < \cdots < \operatorname*{K}_{m}^{n-2} < \operatorname*{K}_{m}^{n} < \operatorname*{K}_{m}^{n-1} < \cdots < \operatorname*{K}_{m}^{m+3} < \operatorname*{K}_{m}^{m+1}$$

for $m \equiv n \pmod 2$, and

$$0 < \operatorname*{K}_{m}^{m} < \operatorname*{K}_{m}^{m+2} < \cdots < \operatorname*{K}_{m}^{n-1} < \operatorname*{K}_{m}^{n} < \operatorname*{K}_{m}^{n-2} < \cdots < \operatorname*{K}_{m}^{m+3} < \operatorname*{K}_{m}^{m+1}$$

for $m \not\equiv n \pmod 2$.

Proof. For any $n > 0$, begin with $a_{n-1} < a_{n-1} +^{-p}(a_n)$, or, equivalently, $0 < \mathrm{K}_{n-1}^{n-1} < \mathrm{K}_{n-1}^{n} < \infty$. It follows that

$$a_{n-2} + {}^{-p}\!\left(\operatorname*{K}_{n-1}^{n-1}\right) > a_{n-2} + {}^{-p}\!\left(\operatorname*{K}_{n-1}^{n}\right) > a_{n-2} > 0 \,,$$

which is to say

$$\operatorname*{K}_{n-2}^{n-1} > \operatorname*{K}_{n-2}^{n} > \operatorname*{K}_{n-2}^{n-2} > 0 \,.$$

[6] The "left-aligned exponent" notation, as well as the English term "continued power," were introduced by T. S. E. Dixon [8] in 1878.

Now repeatedly raise each nonzero quantity to the $-p$th power and add the term of consecutively smaller index. By induction on the index j, one can show that

$$0 < \operatorname*{K}_{n-j}^{n-j} < \operatorname*{K}_{n-j}^{n-j+2} < \cdots < \operatorname*{K}_{n-j}^{n-2} < \operatorname*{K}_{n-j}^{n} < \operatorname*{K}_{n-j}^{n-1} < \cdots < \operatorname*{K}_{n-j}^{n-j-1} < \operatorname*{K}_{n-j}^{n-j+1} .$$

The proof is completed by setting $m = n - j$ and reconciling the parity of the upper and lower indices. □

Two consequences of Proposition 3 are that the even approximants K_0^{2k} form an increasing sequence bounded above by K_0^1, while the odd approximants K_0^{2k+1} form a decreasing sequence bounded below by K_0^0. Thus, the even and odd approximants comprise distinct convergent sequences; when the two limits are identical, the continued $-p$th power converges.

We prefer to state theorems about *divergence*, however, because it is easier to show how the limits of the even and odd approximants may be "pried apart" rather than "forced together." The following definition is comparable to Theorem 2 in [20]. (Note that $\mathrm{K}_0^{2k} < \mathrm{K}_0^{2k+1}$ by Proposition 3.)

Definition 1 (Divergence)**.** *For $p > 0$, a continued $-p$th power of positive terms* diverges *if there exist real numbers c and d such that*

$$\lim_{k\to\infty} \mathrm{K}_{i=0}^{2k+1} = d,\ \lim_{k\to\infty} \mathrm{K}_{i=0}^{2k} = c,\ \textit{and } d - c > 0 .$$

The closed interval $[c,d]$ is the continued $-p$th power's divergence interval*, and we say that K_0^∞* diverges across $[c,d]$.

Proposition 4. *For $p > 0$, a continued $-p$th power diverges if, and only if, one of its tails diverges.*

Proof. This follows directly from the identity

$$\operatorname*{K}_0^n = a_0 + {}^{-p}\big(\ldots + {}^{-p}\big(a_{m-1} + {}^{-p}\Big(\operatorname*{K}_m^n\Big)\big)\ldots\big) .$$

□

Many of the known convergence results for continued pth powers ($p > 0$) are proved by comparing those having arbitrary terms with those having constant terms (see for instance [13, 16, 17]). For $-1 < p < 1$, the continued pth power $\mathrm{K}_0^\infty 1^p$ arises in such comparisons often enough to merit special notation.

Definition 2. *For $-1 < p < 1$ and $n = 0,1,2,\ldots,$*

$$\begin{aligned}
\phi(p) &= \lim_{n\to\infty} \phi_n(p) \\
&= \lim_{n\to\infty} \underbrace{1 + {}^{p}\big(1 + {}^{p}\big(1 + {}^{p}\big(\ldots + {}^{p}(1)\big)\big)\big)}_{n+1 \textit{ terms}} \\
&= 1 + {}^{p}\big(1 + {}^{p}\big(1 + {}^{p}(\ldots)\big)\big) \\
&= \mathrm{K}_{i=0}^{\infty} 1^p .
\end{aligned}$$

We observe that

(1) $\phi(p)$ is well-defined and monotonically increasing for increasing n, as shown in [29, Lemma 1.1 and Corollary 2.4];
(2) $\phi(p)$ is an attracting fixed point of the function $f(x) = 1 + x^p$; and

(3) $\phi(p) > 1$.

Note, too, that $\phi_0(p) = 1$ and $\phi(0) = 2$.

4. A SUFFICIENT DIVERGENCE CONDITION

Theorem 3. *Given* $0 < p < 1$ *and terms* $a_i > 0$, $i = 0, 1, 2, \ldots$, *if*

$$\limsup_{i\to\infty} a_i^{p^i} < 1$$

then $\mathrm{K}_{i=0}^{\infty} a_i^{-p}$ *diverges.*

Proof. If $\limsup_{i\to\infty} a_i^{p^i} = B < 1$, then for every $\varepsilon > 0$ there is a natural number N such that $a_i^{p^i} < B + \varepsilon$ for $i \geq N$. In particular, choose $\varepsilon = \varepsilon_0 > 0$, with corresponding N_0, such that $B + \varepsilon_0 < 1$, and for convenience set $B + \varepsilon_0 = C$. Beginning with $\mathrm{K}_n^n = a_n < C^{q^n}$, where $n > N_0 + j$ and j is a positive integer, we construct finite tails from right to left as follows:

$$
\begin{aligned}
{}^{-p}(a_n) &> C^{-(q^{n-1})} \\
\mathop{\mathrm{K}}_{n-1}^{n} = a_{n-1} + {}^{-p}(a_n) &> C^{-(q^{n-1})} \\
{}^{-p}\Big(\mathop{\mathrm{K}}_{n-1}^{n}\Big) &< C^{q^{n-2}} \\
\mathop{\mathrm{K}}_{n-2}^{n} = a_{n-2} + {}^{-p}\Big(\mathop{\mathrm{K}}_{n-1}^{n}\Big) &< C^{q^{n-2}}(1+1)\,.
\end{aligned}
\tag{9}
$$

Writing $1+1$ as $1 + {}^{p^2}(1) = \phi_1(p^2)$, and suppressing the argument p^2 of the ϕ_i for the time being, we substitute in (9) and continue:

$$
\begin{aligned}
\mathop{\mathrm{K}}_{n-2}^{n} = a_{n-2} + {}^{-p}\Big(\mathop{\mathrm{K}}_{n-1}^{n}\Big) &< C^{q^{n-2}}\phi_2 \\
{}^{-p}\Big(\mathop{\mathrm{K}}_{n-2}^{n}\Big) &> C^{-(q^{n-3})}\ \phi_2^{-p} \\
\mathop{\mathrm{K}}_{n-3}^{n} = a_{n-3} + {}^{-p}\Big(\mathop{\mathrm{K}}_{n-2}^{n}\Big) &> C^{-(q^{n-3})}\ \phi_2^{-p} \\
{}^{-p}\Big(\mathop{\mathrm{K}}_{n-3}^{n}\Big) &< C^{q^{n-4}}\ \phi_2^{p^2} \\
\mathop{\mathrm{K}}_{n-4}^{n} = a_{n-4} + {}^{-p}\Big(\mathop{\mathrm{K}}_{n-3}^{n}\Big) &< C^{q^{n-4}}\ (1 + \phi_2^{p^2}) \\
&= C^{q^{n-4}}\ \phi_3
\end{aligned}
$$

In general, induction on the index j shows that, for j even,

$$\mathop{\mathrm{K}}_{n-j}^{n} < C^{q^{n-j}}\phi_{(j+2)/2}\,, \tag{10}$$

while for j odd,

$$\mathop{\mathrm{K}}_{n-j}^{n} > C^{-(q^{n-j})}\big(\phi_{(j+1)/2}\big)^{-p}\,. \tag{11}$$

Suppose $n = 2k$ for some positive integer k, and let positive integers m and j_0 be such that $m = n - j_0$. Then (10) and (11) respectively become

$$\mathop{\mathrm{K}}_{m}^{2k} < C^{q^m}\phi_{(j_0+2)/2}\,, \quad m \text{ even}, \tag{12}$$

and

$$\mathop{\mathrm{K}}_{m}^{2k} > C^{-(q^m)}\left(\phi_{(j_0+1)/2}\right)^{-p}\,, \quad m \text{ odd}\,. \tag{13}$$

Keeping the same m, but now supposing that $n = 2k-1$ and $m = n - j_1$, the relations (10) and (11) yield, respectively,

$$\mathop{\mathrm{K}}_{m}^{2k-1} > C^{-(q^m)}\left(\phi_{(j_1+1)/2}\right)^{-p}\,, \quad m \text{ even}, \tag{14}$$

and

$$\mathop{\mathrm{K}}_{m}^{2k-1} < C^{q^m}\phi_{(j_1+2)/2}\,, \quad m \text{ odd}\,. \tag{15}$$

Suppose that m is a fixed, even, positive integer. As k goes to infinity in (12) and (14), so do the indices j_0 and j_1, and we have

$$\lim_{k\to\infty}\mathop{\mathrm{K}}_{m}^{2k} \leq C^{q^m}\phi\,, \tag{16}$$

and

$$\lim_{k\to\infty}\mathop{\mathrm{K}}_{m}^{2k-1} \geq C^{-(q^m)}\phi^{-p}\,. \tag{17}$$

We claim that for

$$m > \frac{\log\left(-\dfrac{(p+1)\log\phi}{2\log C}\right)}{\log q} \tag{18}$$

the relations (16) and (17) satisfy the strict inequality

$$\lim_{k\to\infty}\mathop{\mathrm{K}}_{m}^{2k} \leq C^{q^m}\phi < C^{-(q^m)}\phi^{-p} \leq \lim_{k\to\infty}\mathop{\mathrm{K}}_{m}^{2k-1}\,. \tag{19}$$

Observe that the right-hand side of (18) is positive, because $C < 1$, $\phi > 1$, and $q > 1$. From (18) we have

$$\begin{aligned} q^m &> -\frac{(p+1)\log\phi}{2\log C} \\ C^{q^m} &< \phi^{-(p+1)/2} \\ C^{2q^m} &< \frac{1}{\phi^{p+1}} \\ C^{q^m}\phi &< C^{-(q^m)}\phi^{-p}\,. \end{aligned}$$

We conclude that for any positive even integer m satisfying (18), the even and odd tails K_m^{2k} and K_m^{2k-1} converge to separate limits, and K_0^{∞} diverges. A similar argument may be made when m is odd.

□

5. The U-system

We mentioned in Section 3 that a successful tactic in dealing with continued powers of arbitrary terms is to compare them with continued powers of constant terms; we have seen an example of this in the proof of Theorem 3. To obtain a necessary divergence condition, we will compare a continued reciprocal qth root to the following unusual "comparison sequence".

Definition 3. *For $0 < c < d$, $0 < p < 1$, and $q = 1/p$, the U-system $U_p(c,d)$ is the sequence $(u_n)_{n=0}^{\infty}$ defined recursively by*

$$\begin{aligned} u_0 &= c\ , \\ u_1 &= (d-c)^{-q}\ , \text{ and} \\ u_n &= (u_{n-2}^{-q} - u_{n-1})^{-q}\ . \end{aligned}$$

Before proceeding, we must ensure that this definition unfailingly generates positive real numbers.

Proposition 5. *$u_n \in U_p(c,d)$ is a positive real number for each $n = 0,1,2,\dots$ if, and only if,*

$$\frac{d}{c} \geq \phi(p)\ .$$

Proof. Assume first that u_n is a positive real number for $n = 0,1,2,\dots$. Then each difference $u_{n-2}^{-q} - u_{n-1}$ is positive, so in particular

$$\begin{aligned} u_{n-1}^{-q} - u_n &> 0 \\ u_{n-1}^{-q} &> u_n = (u_{n-2}^{-q} - u_{n-1})^{-q} \\ 2u_{n-1} &< u_{n-2}^{-q}\ . \end{aligned}$$

We write the coefficient 2 as $1 + {}^{p}(1) = \phi_1(p)$, suppress the argument p of the ϕ_i, and replace u_{n-1} with its iterative equivalent:

$$\begin{aligned} \phi_1(u_{n-3}^{-q} - u_{n-2})^{-q} &< u_{n-2}^{-q} \\ {}^{-p}\phi_1(u_{n-3}^{-q} - u_{n-2}) &> u_{n-2} \\ u_{n-3}^{-q} &> \phi_2 u_{n-2}\ . \end{aligned}$$

By induction on the index j one may show that

$$u_{n-j}^{-q} > \phi_{j-1} u_{n-j+1}\ .$$

When $j = n$,

$$\begin{aligned} u_0^{-q} = c^{-q} &> \phi_{n-1} u_1 = \phi_{n-1}(d-c)^{-q} \\ c &< {}^{-p}\phi_{n-1}(d-c) \\ c\phi_n &< d\ . \end{aligned}$$

Dividing by c and taking the limit as $n \to \infty$ we have

$$\frac{d}{c} \geq \phi(p)\ .$$

Conversely, assume $d/c \geq \phi(p)$. We show by induction that $u_i > 0$ for $i = 0, 1, 2, \ldots$. First, we certainly have $d/c > 1$ and $d - c > 0$, so the initial case is confirmed by

$$(d-c)^{-q} = u_1 > 0 \, .$$

Now suppose that $u_i > 0$ for $i = 1, 2, \ldots, n$. Again suppressing the argument p of ϕ, we have

$$\begin{aligned}
\frac{d}{c} &\geq \phi > \phi_n \\
(d-c) &> c\phi_{n-1}^p \\
\phi_{n-1}(d-c)^{-q} = \phi_{n-1}u_1 &< c^{-q} = u_0^{-q} \qquad (20) \\
\phi_{n-2}^p u_1 &< u_0^{-q} - u_1 \\
u_1^{-q} &> (u_0^{-q} - u_1)^{-q}\phi_{n-2} = u_2\phi_{n-2} \qquad (21)
\end{aligned}$$

and so on. The inequalities (20) and (21) suggest that, in general,

$$\phi_{n-j}u_j < u_{j-1}^{-q} \, , \qquad (22)$$

and this can be shown by a secondary induction proof on the index j. When $j = n$, we have $\phi_0 = 1$, so (22) becomes

$$u_n < u_{n-1}^{-q} \, ,$$

and hence $u_{n+1} = (u_{n-1}^{-q} - u_n)^{-q} > 0$, which confirms the main induction hypothesis.

□

In the interesting case where $c = d$ in the U-system (irrespective of the roles of these constants as endpoints of a divergence interval), the number $\phi(p)$ turns up unexpectedly and in an ultimately useful way.

Lemma 1. *Set $\phi = \phi(p)$ and define*

$$\begin{aligned}
\theta = \theta(p) &= \phi^{\frac{-p}{p+1}} \\
&= \phi^{\frac{-1}{q+1}} \, . \qquad (23)
\end{aligned}$$

Then θ is a fixed point of $f(x) = (x^{-q} - x)^{-q}$.

Proof. Observe that

$$\begin{aligned}
(\theta^{-q} - \theta)^{-q} &= \left(\phi^{\frac{q}{q+1}} - \phi^{-\frac{1}{q+1}}\right)^{-q} \\
&= \left(\phi^{\frac{q}{q+1}}\left(1 - \phi^{-1}\right)\right)^{-q} \\
&= \phi^{-\frac{q^2}{q+1}}\left(\frac{\phi - 1}{\phi}\right)^{-q} \qquad (24) \\
&= \phi^{-\frac{q^2}{q+1}}\left(\phi^{-q(p-1)}\right) \\
&= \phi^{-\frac{1}{q+1}} = \theta \, .
\end{aligned}$$

At line (24) we applied the identity

$$\frac{\phi - 1}{\phi} = \frac{\phi^p}{\phi} = \phi^{p-1} \, .$$

□

Note that $0 < \theta(p) < 1$ for $0 < p < 1$.

6. A NECESSARY DIVERGENCE CONDITION

We develop our necessary divergence condition in three steps. First, having established the conditions under which $U_p(c,d)$ is a sequence of positive real numbers, in Proposition 6 we use this knowledge to make a term-by-term comparison of the U-system and a divergent continued reciprocal qth root. Next, Proposition 7 sets a growth bound on the elements of the U-system; here the curious number θ from Lemma 1 takes center stage. Finally, in Theorem 4 we show that m can be chosen so that the divergence interval of the tail K_m^∞ is wide enough to allow the previous two results to do their work.

Proposition 6. *Given $0 < p < 1$, suppose that the tail $\mathrm{K}_m^\infty a_i^{-p}$ diverges across $[c,d]$ for some positive integer m. If $d/c \geq \phi(p)$, then for all $j = 0,1,2,\ldots,$*

$$a_{m+j} < u_j \ , \tag{25}$$

where $u_j \in U_p(c,d)$.

Proof. The proof is by induction on j. Suppose m is even. As initial cases for even and odd j, we first have, trivially,

$$\mathop{\mathrm{K}}_{m}^{m} = a_m < c = u_0 \ ,$$

but also

$$\begin{aligned} d < \mathop{\mathrm{K}}_{m}^{m+1} &= a_m + {}^{-p}(a_{m+1}) \\ d - c < d - a_m &< {}^{-p}(a_{m+1}) \\ u_1 = (d-c)^{-q} &> a_{m+1} \ . \end{aligned}$$

Now suppose that (25) holds for $j = 0,1,\ldots,n-1$, where n is even. We "unwrap" K_m^{m+n}, applying (25) as follows:

$$\begin{aligned} \mathop{\mathrm{K}}_{m}^{m+n} = a_m + {}^{-p}\Big(\mathop{\mathrm{K}}_{m+1}^{m+n}\Big) &< c \\ {}^{-p}\Big(\mathop{\mathrm{K}}_{m+1}^{m+n}\Big) &< c = u_0 \\ \mathop{\mathrm{K}}_{m+1}^{m+n} = a_{m+1} + {}^{-p}\Big(\mathop{\mathrm{K}}_{m+2}^{m+n}\Big) &> u_0^{-q} \\ {}^{-p}\Big(\mathop{\mathrm{K}}_{m+2}^{m+n}\Big) &> u_0^{-q} - u_1 \quad \text{(because } a_{m+1} < u_1\text{)} \\ \mathop{\mathrm{K}}_{m+2}^{m+n} = a_{m+2} + {}^{-p}\Big(\mathop{\mathrm{K}}_{m+3}^{m+n}\Big) &< (u_0^{-q} - u_1)^{-q} = u_2 \ . \end{aligned}$$

Continuing in this way, a secondary induction on the index $i = 0,1,2,\ldots$ shows that

$$\mathop{\mathrm{K}}_{m+i}^{m+n} < u_i$$

for i even. Thus, when $i = n$ we have

$$\mathop{\mathrm{K}}_{m+n}^{m+n} = a_{m+n} < u_n \ ,$$

and the induction on j is complete. These constructions can be modified accordingly for m odd, per Proposition 3. □

Proposition 7. *Given $0 < p < 1$, $q = 1/p$, and positive real numbers c and d. If $d/c \geq \phi = \phi(p)$, then for each $u_i \in U_p(c,d)$, $i = 0,1,2,\ldots$,*

$$u_i \leq \theta \left(\frac{c}{\theta}\right)^{(-q)^i} . \tag{26}$$

Proof. By induction on i. The initial cases $i = 0$ and $i = 1$ are verified by

$$u_0 = c = \theta \left(\frac{c}{\theta}\right)^{(-q)^0}$$

and

$$\begin{aligned} u_1 = (d-c)^{-q} &\leq (c(\phi - 1))^{-q} \\ &= c^{-q}\phi^{-1} \\ &= \theta \left(\frac{c}{\theta}\right)^{(-q)^1} . \end{aligned}$$

Suppose that (26) holds for $i = 0,1,\ldots,n$. Then

$$\begin{aligned} u_{n-1}^{-q} &\geq \theta^{-q} \left(\frac{c}{\theta}\right)^{(-q)^n} \\ -u_n &\geq -\theta \left(\frac{c}{\theta}\right)^{(-q)^n} \end{aligned}$$

and hence

$$\begin{aligned} u_{n+1} = (u_{n-1}^{-q} - u_n)^{-q} &\leq \left(\theta^{-q} \left(\frac{c}{\theta}\right)^{(-q)^n} - \theta \left(\frac{c}{\theta}\right)^{(-q)^n} \right)^{-q} \\ &= \left(\theta^{-q} - \theta\right)^{-q} \left(\frac{c}{\theta}\right)^{(-q)^{n+1}} \\ &= \theta \left(\frac{c}{\theta}\right)^{(-q)^{n+1}} , \end{aligned}$$

where in the last step we applied Lemma 1. This verifies the induction hypothesis for $i = n+1$. □

The proof of the following lemma is left to the reader.

Lemma 2. *For $q > 1$ and real numbers a, x, and y satisfying $0 < a < x < y$,*

$$\frac{y}{x} < \left(\frac{y}{x}\right)^q < \left(\frac{y-a}{x-a}\right)^q .$$

We are at last in a position to prove our necessary condition for divergence.

Theorem 4. *Suppose that* $\mathrm{K}_{i=0}^{\infty} a_i^{-p}$ *diverges across* $[c,d]$ *for* $0 < p < 1$. *Then there exists an* $m \geq 0$ *such that either*

$$a_{m+2i}^{p^{2i}} < 1 \quad or \quad a_{m+2i+1}^{p^{2i+1}} < 1 \tag{27}$$

for $i = 0, 1, 2, \ldots$.

Proof. Let $c_0 = c$ and $d_0 = d$. We "unwrap" the even and odd approximants by subtracting terms and raising the remaining quantities to the $-q$ power, at each step labeling the new quantities as follows:

$$\lim_{k\to\infty} \mathop{\mathrm{K}}_{i=0}^{2k} = c_0 < d_0 = \lim_{k\to\infty} \mathop{\mathrm{K}}_{i=0}^{2k+1}$$

$$\lim_{k\to\infty} \mathop{\mathrm{K}}_{i=1}^{2k} = (c_0 - a_0)^{-q} = c_1 > d_1 = (d_0 - a_0)^{-q} = \lim_{k\to\infty} \mathop{\mathrm{K}}_{i=1}^{2k+1}$$

$$\lim_{k\to\infty} \mathop{\mathrm{K}}_{i=2}^{2k} = (c_1 - a_1)^{-q} = c_2 < d_2 = (d_1 - a_1)^{-q} = \lim_{k\to\infty} \mathop{\mathrm{K}}_{i=2}^{2k+1}$$

and in general, for even j,

$$\lim_{k\to\infty} \mathop{\mathrm{K}}_{i=j}^{2k} = (c_{j-1} - a_{j-1})^{-q} = c_j < d_j = (d_{j-1} - a_{j-1})^{-q} = \lim_{k\to\infty} \mathop{\mathrm{K}}_{i=j}^{2k+1}$$

while for odd j,

$$\lim_{k\to\infty} \mathop{\mathrm{K}}_{i=j}^{2k} = (c_{j-1} - a_{j-1})^{-q} = c_j > d_j = (d_{j-1} - a_{j-1})^{-q} = \lim_{k\to\infty} \mathop{\mathrm{K}}_{i=j}^{2k+1}$$

Assume j is even. We show that

$$\lim_{j\to\infty} \frac{d_j}{c_j} = \infty\,.$$

Applying Lemma 2 repeatedly, we have

$$\begin{aligned}
\frac{d_j}{c_j} &= \left(\frac{d_{j-1} - a_{j-1}}{c_{j-1} - a_{j-1}}\right)^{-q} = \left(\frac{c_{j-1} - a_{j-1}}{d_{j-1} - a_{j-1}}\right)^{q} \\
&> \left(\frac{c_{j-1}}{d_{j-1}}\right)^{q} = \left(\frac{d_{j-2} - a_{j-2}}{c_{j-2} - a_{j-2}}\right)^{q^2} \\
&> \left(\frac{d_{j-2}}{c_{j-2}}\right)^{q^2} = \left(\frac{c_{j-3} - a_{j-3}}{d_{j-3} - a_{j-3}}\right)^{q^3} \,,
\end{aligned}$$

and induction on i shows that, in general,

$$\frac{d_j}{c_j} > \begin{cases} \left(\dfrac{d_{j-i}}{c_{j-i}}\right)^{q^i} & \text{for } i \text{ even}\,; \\[2ex] \left(\dfrac{c_{j-i}}{d_{j-i}}\right)^{q^i} & \text{for } i \text{ odd}\,. \end{cases}$$

Thus, when $i = j$ we have

$$\frac{d_j}{c_j} > \left(\frac{d_0}{c_0}\right)^{q^j}\,,$$

and because $d_0/c_0 > 1$, it follows that $\lim_{j\to\infty} d_j/c_j = \infty$. Similarly, when $i = j-1$, we obtain

$$\frac{d_j}{c_j} > \left(\frac{c_1}{d_1}\right)^{q^{j-1}},$$

and $c_1/d_1 > 1$ again implies $\lim_{j\to\infty} d_j/c_j = \infty$. An entirely parallel argument applies if j is assumed to be odd.

We have established that divergence intervals for successive tails increase in length without bound. It follows that there exists a positive even integer m such that

$$\lim_{k\to\infty} \mathop{\mathrm{K}}_{m}^{2k} = c_m < d_m = \lim_{k\to\infty} \mathop{\mathrm{K}}_{m}^{2k+1}$$

and for which $d_m/c_m > \phi(p)$. Propositions 6 and 7 then imply that for $\nu = 0, 1, 2, \ldots$,

$$a_{m+\nu} < u_\nu \le \theta\left(\frac{c}{\theta}\right)^{(-q)^\nu}. \tag{28}$$

Recall now that $\theta < 1$. The parity of ν, combined with the options $c/\theta \le 1$ and $c/\theta > 1$, allows for four interpretations of (28). Two cases of interest will complete the proof.

(1) $c/\theta \le 1$ and $\nu = 2i$, $i = 0, 1, 2, \ldots$. In this case (28) yields

$$a_{m+2i} < \theta\left(\frac{c}{\theta}\right)^{q^{2i}}$$
$$a_{m+2i}^{p^{2i}} < \theta^{p^{2i}}\frac{c}{\theta} < 1\,.$$

(2) $c/\theta > 1$ and $\nu = 2i+1$, $i = 0, 1, 2, \ldots$. Here we have

$$a_{m+2i+1} < \theta\left(\frac{c}{\theta}\right)^{-q^{2i+1}} = \theta\left(\frac{\theta}{c}\right)^{q^{2i+1}}$$
$$a_{m+2i+1}^{p^{2i+1}} < \theta^{p^{2i+1}}\frac{\theta}{c} < 1\,.$$

□

Theorem 1 now follows directly from Theorems 3 and 4, inasmuch as both of the necessary relations in (27) are satisfied by the sufficient relation

$$\limsup_{i\to\infty} a_i^{p^i} < 1\,.$$

7. Examples

We conclude by examining a few sequences $(a_i)_{i=0}^\infty$ which illustrate aspects of Theorems 1, 3, and 4.

(1) $a_i = 1$. The continued reciprocal qth root with these terms is convergent, of course, being nothing more than $\phi(p)$ (Definition 2). Indeed, Laugwitz showed in [20] that a continued reciprocal qth root with constant terms $\alpha > 0$ converges, and Theorem 1 confirms this, since $\limsup_{i\to\infty} \alpha^{p^i} = 1$.

(2) $a_i = \beta^{-(q^i)}$, $\beta > 1$. These terms satisfy Theorem 1:

$$\beta_i^{p^i} = (\beta^{-(q^i)})^{p^i} = \frac{1}{\beta} < 1\,,$$

thus $\mathrm{K}_{i=0}^{\infty} a_i^p$ diverges. For instance, when $p = \frac{1}{2}$, the values $\beta = 1.2$, 1.4, and 1.6 yield the approximant values $y = \mathrm{K}_0^n$, $n = 0, 1, 2, \ldots, 16$ graphed in Figure 1.

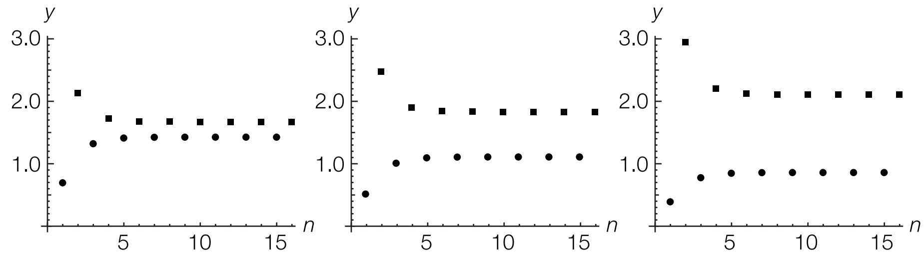

FIGURE 1. Values of odd approximants (square dots) and even approximants (round dots) of three divergent continued reciprocal square roots ($p = \frac{1}{2}$) using the terms a_i in Example 2. Left, $\beta = 1.2$; center, $\beta = 1.4$; right, $\beta = 1.6$.

(3) $a_i = \gamma^{(-q)^i}$, $\gamma > 1$. Here we find

$$\gamma_i^{p^i} = (\gamma^{(-q)^i})^{p^i} = \gamma^{(-1)^i} \,.$$

In this case $\limsup_{i\to\infty} a_i^{p^i} = \gamma > 1$, and the continued reciprocal qth root with these terms converges. This can also be seen by multiplying $\phi(p)$ by a fraction having identical numerator and denominator γ, and distributing either the numerator or denominator to the right:

$$\begin{aligned}\phi(p) &= \frac{\gamma}{\gamma}\Big(1 + {}^p\Big(1 + {}^p\Big(1 + {}^p\Big(1 + {}^p\Big(\ldots\Big)\Big)\Big)\Big)\Big)\\ &= \frac{1}{\gamma}\Big(\gamma + {}^p\Big(\gamma^{-\frac{1}{p}} + {}^p\Big(\gamma^{\frac{1}{p^2}} + {}^p\Big(\gamma^{-\frac{1}{p^3}} + {}^p\Big(\ldots\Big)\Big)\Big)\Big)\Big)\\ &= \gamma\Big(\gamma^{-1} + {}^p\Big(\gamma^{\frac{1}{p}} + {}^p\Big(\gamma^{-\frac{1}{p^2}} + {}^p\Big(\gamma^{\frac{1}{p^3}} + {}^p\Big(\ldots\Big)\Big)\Big)\Big)\Big)\,.\end{aligned}$$

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