Similarly, since the vertical chains W_1 and W_2 do not meet, the number of columns n must be greater than 2.

We now consider the $(m-2)\times(n-2)$ board (See FIGURE 1) obtained by deleting rows 1 and m and columns 1 and n. In this board we apply the inductive hypothesis in its equivalent form: there must be either a white chain from column 2 to column $n-1$ or a black chain from row 2 to row $m-1$. We assume, without loss of generality, the former. This chain W_3 must intersect chains W_1 and W_2, so W_1, W_2 and W_3 together form a white chain from row 1 to row m. This completes the proof.

We note, in conclusion, that this proof can easily be modified to deal with other games of this sort, for instance, Bridge-it.

Reference

[1] Anatole Beck, Michael Bleicher, and Donald Crowe, Excursions into Mathematics, Worth, New York, 1969.

A Double Butterfly Theorem

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To the extensive annals of geometric lepidopterology we add a further modification of the well-known butterfly problem. Let us define a "**butterfly**," denoted by $)B($, as the two triangles formed by the diagonals and two opposite sides of a convex quadrilateral, and refer to these triangles as

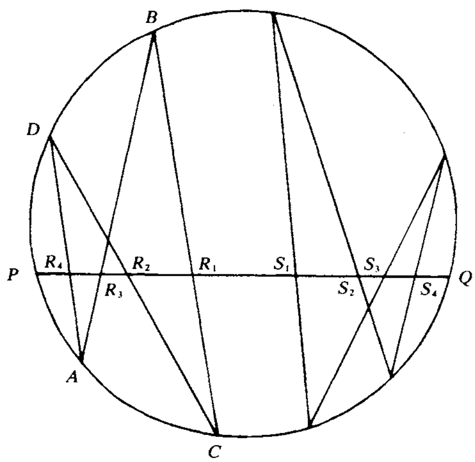

FIGURE 1.

"wings" of the butterfly. FIGURE 1, depicting two butterflies inscribed in a circle, illustrates our main result:

THEOREM. *Let PQ be a fixed chord of a circle. Let $)R($ and $)S($ be inscribed in the circle and oriented such that their wings cut PQ (in order from left to right) at R_4, R_3, R_2, R_1, and S_1, S_2, S_3, S_4, respectively. If $PR_1 = QS_1$, $PR_2 = QS_2$, and $PR_3 = QS_3$, then $PR_4 = QS_4$.*

Proof. Consider $)R($. Denoting by $(UVWX)$ the double ratio on points U, V, W, and X, we have

$$(PR_4R_3Q) = \frac{\sin\angle PAB}{\sin\angle BAD} \div \frac{\sin\angle PAQ}{\sin\angle QAD}, \qquad (PR_2R_1Q) = \frac{\sin\angle PCB}{\sin\angle BCD} \div \frac{\sin\angle PAQ}{\sin\angle QCD}.$$

Since angles inscribed in the same arc are equal, we deduce that $(PR_4R_3Q) = (PR_2R_1Q)$. Expanding and cancelling, we have

$$\text{(1)} \qquad \frac{PR_3 \cdot QR_4}{R_3R_4} = \frac{PR_1 \cdot QR_2}{R_1R_2}.$$

Identical reasoning for $)S($ establishes that $(QS_4S_3P) = (QS_2S_1P)$, by which we obtain

$$\text{(2)} \qquad \frac{QS_3 \cdot PS_4}{S_3S_4} = \frac{QS_1 \cdot PS_2}{S_1S_2}.$$

Since in the given conditions $QS_1 = PR_1$, $PS_2 = QR_2$, and $S_1S_2 = R_1R_2$, substitution in (2) gives us $QS_3 \cdot PS_4 \colon S_3S_4 = PR_1 \cdot QR_2 \colon R_1R_2$ which with (1) yields $QS_3 \cdot PS_4 \colon S_3S_4 = PR_3 \cdot QR_4 \colon R_3R_4$ or, since $QS_3 = PR_3$,

$$\text{(3)} \qquad PS_4 \cdot R_3R_4 = QR_4 \cdot S_3S_4.$$

Now, $PS_4 = PS_3 + S_3S_4$ and $QR_4 = QR_3 + R_3R_4$. Substituting these into (3) yields

$$PS_3 \cdot R_3R_4 + S_3S_4 \cdot R_3R_4 = QR_3 \cdot S_3S_4 + R_3R_4 \cdot S_3S_4.$$

Cancelling the identical terms and taking into account the fact, evident from the given conditions, that $PS_3 = QR_3$, we have $R_3R_4 = S_3S_4$ which by subtraction gives us the result $PR_4 = QS_4$.

The reader may verify various special cases of the theorem which occur when, in addition to the given conditions, (a) $R_3 = R_2$ and $S_3 = S_2$, (b) $R_1 = S_1$, (c) $R_3 = R_2 = S_1$ and $S_3 = S_2 = R_1$, (d) all intersection points are the same, which is a reduction to the original butterfly problem, or (e) $R_3 = R_2 = R_1 = S_1 = S_2 = S_3$, which produces Klamkin's extension of the problem [**1**].

Reference

[1] M. S. Klamkin, An extension of the butterfly problem, this MAGAZINE, 38 (1965) 206–208.

The Arithmetic Mean–Geometric Mean Inequality: A New Proof

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In this note, we give a simple inductive proof for the arithmetic mean–geometric mean inequality. Other inductive proofs already known can be found in [**1**, §5, pp. 4–5, §11, pp. 9–10, and §13, pp. 11–12], [**2**, p. 46], [**3**, §2.6, pp. 18–21], [**4**] and [**5**, pp. 285–286].

The inequality concerned is

$$A = \frac{a_1 + a_2 + \cdots + a_n}{n} \geqq \sqrt[n]{a_1a_2\cdots a_n} = G,$$

where $a_1, a_2, \ldots, a_n$ are positive numbers. There is equality if and only if $a_1 = a_2 = \cdots = a_n$.

Suppose that $a_1 \geqq a_2 \geqq \cdots \geqq a_n$. Then, clearly, $a_1 \geqq G \geqq a_n$, with equality if and only if $a_1 = a_2 = \cdots = a_n$, and so

$$\text{(1)} \qquad a_1 + a_n - \left(G + \frac{a_1a_n}{G}\right) = \frac{1}{G}(a_1 - G)(G - a_n) \geqq 0.$$