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Families of maximal perigees

By Dixon J. Jones

Abstract. Previously we investigated the cycle structure of the iterated function system comprising the maps $f(x) = ax + 1$ and $g(x) = bx$, with $a > 0$ and $0 < b < 1$. We defined certain equivalence classes of periodic orbits based on their perigee representations, and we exhibited an explicit expression corresponding to the maximal perigee in each class. Here we show how maximal perigees develop across classes as the cycle length increases. This leads in a natural way to the organization of maximal perigees into *families* and their component *branches*. We then derive a rational function that attains all maximal perigee values in a given branch.

In [3] we investigated the cycle structure of the iterated function system $\Psi(a, b)$ defined on the real numbers by the maps $f(x) = ax + 1$ and $g(x) = bx$, where $a > 0$ and $0 < b < 1$. Taking a combinatorial approach, we considered the finite compositions, or *words*, in f and g that define cycles in $\Psi(a, b)$. For instance, in $\Psi\left(3, \frac{1}{2}\right)$, the word $gfggff$ corresponds to the 6-cycle point $x_1 = -\frac{16}{19}$, which is the solution to $gfggff(x_1) = x_1$; indeed, $-\frac{16}{19}$ is the *perigee*, or point of least magnitude, of its cycle, and $gfggff$ is the cycle's *perigee word*. For convenience, we recall some notation and definitions from [3].

Definition 1. *Let w be a word of n letters chosen from the set $\{f, g\}$.*

a: *The f-rank of w, denoted by r, is its number of fs.* (Because words of the form g^n yield just the trivial cycle point 0, we assume here that $r \in \{1, \ldots, n\}$.)

b: *The density, denoted by α, is the ratio $(n - r)/r$ of the number of gs to fs in w.*

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c: *The base markers in w are the fs indexed from left to right and from 1 to r in w.*

d: *For $i \in \{1, \ldots, r\}$, the ith gap d_i is the number of gs between base markers f_{i-1} and f_i, indices taken modulo r.* (It is helpful to define $d_0 = 0$ for some calculations.) *The ordered r-tuple $D(w) = (d_1, d_2, \ldots, d_r)$ is the gaps vector of w.*

e: *The g-rank of base marker f_i is the number of gs to its left in w, and is denoted by q_i. Equivalently, it is the sum of gaps $d_1+d_2+\cdots+d_i$. The ordered r-tuple $Q(w) = (q_1, q_2, \ldots, q_r)$ is the cycle word code, or, more briefly, the code of w.*

f: *F_r^n, $r \in \{1, \ldots, n\}$, is the set of all n-letter words with f-rank r and with f the rightmost letter.*

g: *P_r^n is the subset of F_r^n that contains precisely those words corresponding to cycle perigees.* (P_r^n is an equivalence class in the set of perigee words of length n; two such words are related if they have the same f-rank, or, equivalently, if they have the same density.)

The *maximal perigee word* $w_{\max} \in P_r^n$ is the word corresponding to the perigee of greatest magnitude in P_r^n. The following result from [3] characterizes this special word.

Maximal Perigee Property. *In $\Psi(a, b)$, the maximal perigee word $w_{\max} \in P_r^n$ with density α has code*

$$Q(w_{\max}) = (\lceil \alpha \rceil, \lceil 2\alpha \rceil, \ldots, \lceil r\alpha \rceil) \ ,$$

where $\lceil \cdot \rceil$ is the ceiling function.

The proof makes use of an extension of Chisala's Lemma [1, Lemma 3.1]. In what follows, we partition gaps vectors into what we will call *Chisala blocks* (such blocks are used in the proof of Chisala's original lemma): If $D(w) = (d_1, \ldots, d_r)$ and the density of w is α, then the Chisala block form of D is

$$(d_1, \ldots, d_{i_1-1} \mid d_{i_1}, \ldots, d_{i_2-1} \mid \ \ldots \ \mid d_{i_\ell}, \ldots, d_r) \ ,$$

where the vertical bars mark the jumps from terms $d_{i_j-1} < \alpha$ to terms $d_{i_j} \geq \alpha$. The terms between each sequential pair of bars comprise the Chisala blocks $B_1, \ldots, B_\ell$; that is,

$$B_j = (d_{i_j}, \ldots, d_{i_{j+1}-1}), \ j \in \{1, \ldots, \ell - 1\} \ ,$$

and

$$B_\ell = (d_{i_\ell}, \ldots, d_r, d_1, \ldots, d_{i_1-1}) \; .$$

For instance, if $D(w) = (0,0,1,1,0,1,0,1,0,0,1,1)$, then $\alpha = \frac{1}{2}$, and the Chisala block form is

$$D(w) = (0,0 \mid 1,1,0 \mid 1,0 \mid 1,0,0 \mid 1,1) \; ,$$

where the last block is $B_4 = (1,1,0,0)$.

In this paper we use Chisala blocks and computations with floor and ceiling functions to completely describe the growth of maximal perigee words across the sets P_r^n as n increases. This leads to a natural organization of perigee words into *families* and their component *branches*, and allows us to derive a rational function attaining all maximal perigee values in a given branch.

1. Maximal families

Table 1 shows the maximal perigee words in P_{i+4}^{2i+4}, $i \in \{0, \ldots, 8\}$. Each word has one more g and one more f than the word preceding it. The code of each word satisfies the Maximal Perigee Property for its particular density $\alpha = i/(i{+}4)$. Our first goal is to reveal a pattern by which this and similar tables can be extended, but without recourse to the Maximal Perigee Property for each new word length n and f-rank r. We denote the nonnegative integers by $\mathbb{Z}_+$.

Table 1. Maximal perigee words in P_{i+4}^{2i+4}, $i \in \{0, \ldots, 8\}$.

n	r	Maximal perigee word in P_r^n
4	4	$ffff$
6	5	$gfffff$
8	6	$gfffgfff$
10	7	$gffgffgfff$
12	8	$gffgffgffgff$
14	9	$gfgffgffgffgff$
16	10	$gfgffgffgfgffgff$
18	11	$gfgffgfgffgfgffgff$
20	12	$gfgffgfgffgfgffgfgff$

Definition 2. *In* $\Psi(a,b)$,

a: *for* $m \in \mathbb{Z}_+$*, the* m*th maximal family* $\mathcal{A}^m$ *is the set of maximal perigee words in* P^{2i+m}_{i+m} *over all* $i \in \mathbb{Z}_+$ *and with* i *and* m *not both* 0*;*

b: *for* $m \in \mathbb{N}$ *and* $k \in \{1, \dots, m\}$*, the* k*th branch of the* m*th maximal family, denoted by* $\mathcal{A}^m_k$*, is the set of maximal perigee words from* $P^{(2j-1)m+2k}_{jm+k}$ *over all* $j \in \mathbb{Z}_+$*.*

Table 1 shows the first nine members of $\mathcal{A}^4$.

Since $\{(2j-1)m+2k, jm+k\}$ is of the form $\{2i+m, i+m\}$ for $i = (j-1)m+k$, it is clear that $\mathcal{A}^m_k \subset \mathcal{A}^m$ and that

$$\bigcup_{k \in \{1, \dots, m\}} \mathcal{A}^m_k = \mathcal{A}^m .$$

Note that the density of any $w \in \mathcal{A}^m_k$ is

$$\frac{(2j-1)m + 2k - (jm+k)}{jm+k} = \frac{(j-1)m+k}{jm+k} < 1 . \tag{1}$$

Using the Maximal Perigee Property, it may easily be verified that in P^{2i}_i, $w_{\max} = (gf)^i$. Therefore we have

$$\mathcal{A}^0 = \{(gf)^i \,, i \geq 1\} .$$

A more illuminating example is $\mathcal{A}^4$. While the first nine members of this set are listed in Table 1, the pattern of growth is shown in more detail in Table 2, where the columns under the four values of k represent the branches $\mathcal{A}^4_k$. We write the first word in the list, $ffff$, in its base marker form $f_1 f_2 f_3 f_4$ (Definition 1c). We see that successive words in $\mathcal{A}^4$ are created by inserting the subword gf to the left of these initial base markers in a certain order. (The index j, to be used in an upcoming definition, is the highest repetition number of the subword gf occurring in the word.) The insertion order is cyclic modulo 4, and proceeds as follows:

$k = 1$**:** Insert gf to the left of

$$f_1 = f_{1+\lfloor 0 \cdot 4/1 \rfloor} .$$

$k = 2$**:** Insert gf to the left of

$$f_1 = f_{1+\lfloor 0 \cdot 4/2 \rfloor} \text{ and}$$
$$f_3 = f_{1+\lfloor 1 \cdot 4/2 \rfloor} .$$

$k = 3$**:** Insert gf to the left of

$$f_1 = f_{1+\lfloor 0\cdot 4/3 \rfloor} ,$$
$$f_2 = f_{1+\lfloor 1\cdot 4/3 \rfloor} , \text{ and}$$
$$f_3 = f_{1+\lfloor 2\cdot 4/3 \rfloor} .$$

$k = 4$**:** Insert gf to the left of

$$f_1 = f_{1+\lfloor 0\cdot 4/4 \rfloor} ,$$
$$f_2 = f_{1+\lfloor 1\cdot 4/4 \rfloor} ,$$
$$f_3 = f_{1+\lfloor 2\cdot 4/4 \rfloor} , \text{ and}$$
$$f_4 = f_{1+\lfloor 3\cdot 4/4 \rfloor} .$$

Table 2. The first nine members of $\mathcal{A}^4$, arranged to emphasize the pattern of development. $C = gf$. The four rightmost columns are the four branches $\mathcal{A}^4_k$. The index j is the highest repetition number of C occurring in the word.

n	r	j	$k = 1$	$k = 2$	$k = 3$	$k = 4$
4	4	0				$f_1f_2f_3f_4$
6	5	1	$Cf_1f_2f_3f_4$			
8	6	1		$Cf_1f_2Cf_3f_4$		
10	7	1			$Cf_1Cf_2Cf_3f_4$	
12	8	1				$Cf_1Cf_2Cf_3Cf_4$
14	9	2	$C^2f_1Cf_2Cf_3Cf_4$			
16	10	2		$C^2f_1Cf_2C^2f_3Cf_4$		
18	11	2			$C^2f_1C^2f_2C^2f_3Cf_4$	
20	12	2				$C^2f_1C^2f_2C^2f_3C^2f_4$

In the next section we make some general definitions based on this example. Ultimately, for each $k \in \{1, \ldots, m\}$ we will derive a rational function that generates the perigees of all cycles in the kth branch of the mth family.

2. Active markers and uniform extensions

Definition 3. *Given $w \in F^n_r$ with base markers $f_1, f_2, \ldots, f_r$, and given $k \in \{1, \ldots, r\}$.*

a: *An integer $i \in \{1, \ldots, r\}$ is active mod k if*

$$i = 1 + \left\lfloor (\ell - 1)\frac{r}{k} \right\rfloor$$

for some $\ell \in \{1, \ldots, k\}$, and is inactive mod k otherwise. Note that $i = 1$ is always active mod k for any $k \in \{1, \ldots, r\}$. If $r = 5$, then 1, 2, and 4 are active mod 3, while 3 and 5 are inactive mod 3.

b: *The active markers mod k in w are the k base markers f_{i_ℓ} where i_ℓ is active mod k for each $\ell \in \{1, \ldots, k\}$.* We indicate the active markers within a word by placing dots over the appropriate base markers, and the representation of w showing its active markers mod k is abbreviated as $\langle w \rangle_k$. Thus,

$$\langle gffgfff \rangle_3 = g\dot{f}_1\dot{f}_2 g f_3 \dot{f}_4 f_5 \ .$$

c: *The activation vector for $\langle w \rangle_k$ is the r-tuple $E\langle w \rangle_k = (e_1, \ldots, e_r)$, where*

$$e_i = \begin{cases} 1 & \textit{if } i \textit{ is active mod } k, \\ 0 & \textit{otherwise.} \end{cases}$$

To continue the example of part (b), $E\langle gffgfff \rangle_3 = (1, 1, 0, 1, 0)$.

d: *The ith activation counter for $\langle w \rangle_k$ is*

$$\kappa_i = \sum_{\nu=1}^{i} e_\nu \ .$$

The vector of activation counters for the example of parts (b) and (c) above is $(1, 2, 2, 3, 3)$. Note that $\kappa_r = k$.

e: *For any $j \in \mathbb{N}$, the jth uniform extension mod k of w, written $U_j\langle w \rangle_k$, is the word formed by inserting $(gf)^j$ to the immediate left of every active marker mod k in w, and inserting $(gf)^{j-1}$ to the immediate left of every inactive marker. We also define $U_0\langle w \rangle_k = \langle w \rangle_k$.* For example, using $w \in F_5^7$ from (b) above, we have

$$\begin{aligned} U_2\langle w \rangle_3 &= g(gf)^2\dot{f}_1(gf)^2\dot{f}_2 g(gf)f_3(gf)^2\dot{f}_4(gf)f_5 \\ &= ggfgffgfgffggffgfgffgff \ . \end{aligned}$$

In the remainder of this section we show how active markers and uniform extensions are involved in the development of maximal perigee words. The main result is that the kth branch of the mth maximal family is precisely the set of uniform extensions mod k of f^m.

Theorem 1. *Given $m, j \in \mathbb{N}$, and $k \in \{1, \ldots, m\}$, the jth element of $\mathcal{A}_k^m$ is $U_j\langle f^m\rangle_k$.*

We will prove Theorem 1 by showing that the code for $U_j\langle f^m\rangle_k$ satisfies the Maximal Perigee Property in $P_{jm+k}^{(2j-1)m+2k}$. The key fact is that, for a fixed k, the activation vector $E\langle f^m\rangle_k$ determines the Chisala block form of a uniform extension's gaps vector. We observe that $E\langle f^m\rangle_k$ has m terms, which are the "seeds" of the m Chisala blocks in $U_j\langle f^m\rangle_k$. Table 3 illustrates this for $\mathcal{A}_2^4$ (the column "$k = 2$" in Table 2).

Table 3. Development of $\mathcal{A}_2^4$. The first row shows $\langle ffff\rangle_2 = \dot{f}_1 f_2 \dot{f}_3 f_4$ (not a member of $\mathcal{A}_2^4$), with its activation vector in boldface. The remainder of the table lists the first three members of $\mathcal{A}_2^4$, with their gaps vectors shown in Chisala block form.

n	r	Maximal perigee word in P_r^n	Vector
4	4	$\dot{f}_1 f_2 \dot{f}_3 f_4$	$(\mathbf{1}\ ,\ \mathbf{0}\ ,\ \mathbf{1}\ ,\ \mathbf{0})$
8	6	$(gf)\dot{f}_1 f_2 (gf)\dot{f}_3 f_4$	$(1, 0 \mid 0 \mid 1, 0 \mid 0)$
16	10	$(gf)^2\dot{f}_1(gf)f_2(gf)^2\dot{f}_3(gf)f_4$	$(1, 1, 0 \mid 1, 0 \mid 1, 1, 0 \mid 1, 0)$
24	14	$(gf)^3\dot{f}_1(gf)^2 f_2(gf)^3\dot{f}_3(gf)^2 f_4$	$(1, 1, 1, 0 \mid 1, 1, 0 \mid 1, 1, 1, 0 \mid 1, 1, 0)$

We approach the proof through several intermediate results involving calculations with floor and ceiling functions. We begin by recalling that, for real numbers M and N,

$$\left\lfloor \frac{M}{N} \right\rfloor = \frac{M}{N} - \frac{M \bmod N}{N}\ , \tag{2}$$

as defined in [2], while

$$\left\lceil \frac{M}{N} \right\rceil = \frac{M}{N} + \frac{(-M) \bmod N}{N} \tag{3}$$

is an apparently lesser-known companion formula.[1] We begin with a lemma that gives two simple ways to calculate activation counters.

Lemma 1. *Given $m \in \mathbb{N}$ and $k \in \{1, \ldots, m\}$, let $i_\ell \in \{1, \ldots, m\}$ be active mod k for each $\ell \in \{1, \ldots, k\}$. Then for each $i \in \{i_\ell, \ldots, i_{\ell+1} - 1\}$ the ith*

[1]Equation (3) is listed in the addenda for [2, page 83] on the web site http://www-cs-faculty.stanford.edu/~knuth/gkp.html.

activation counter for $\langle f^m \rangle_k$ *is*

$$\kappa_i = \left\lceil \frac{ik}{m} \right\rceil = \ell \; . \tag{4}$$

PROOF. It should first be noted that when $\ell = k$,

$$i_{\ell+1} - 1 = 1 + \left\lfloor (k)\frac{m}{k} \right\rfloor - 1 = m \; .$$

The numbers $\{i_\ell, \ldots, i_{\ell+1} - 1\}$, taken as indices of terms in the activation vector $E\langle f^m \rangle_k$, specify strings

$$(e_{i_\ell}, \ldots, e_{i_{\ell+1}-1}), \quad \ell \in \{1, \ldots, k\} \; ,$$

which contain either a single element $e_{i_\ell} = 1$, or else $e_{i_\ell} = 1$ and every other term is 0. Thus, if $i \in \{i_\ell, \ldots, i_{\ell+1} - 1\}$,

$$\kappa_i = \sum_{\nu=1}^{i} e_\nu = \sum_{\nu \in \{i_1, \ldots, i_2 - 1\}} e_\nu + \sum_{\nu \in \{i_2, \ldots, i_3 - 1\}} e_\nu + \cdots + \sum_{\nu = i_\ell}^{i} e_\nu = \ell \; ,$$

since the first $\ell - 1$ complete strings and the beginning of the last string all contribute 1 to the total. Now, $i \in \{i_\ell, \ldots, i_{\ell+1} - 1\}$ means

$$1 + \left\lfloor (\ell - 1)\frac{m}{k} \right\rfloor \le i \le \left\lfloor \ell \frac{m}{k} \right\rfloor \; .$$

Apply formula (2), then multiply throughout by k/m to obtain

$$\ell - 1 + \frac{k - (\ell - 1)m \bmod k}{m} \le \frac{ik}{m} \le \ell - \frac{\ell m \bmod k}{m} \; . \tag{5}$$

For any integer N, we have $N \bmod k \le k - 1$, so

$$\frac{1}{m} \le \frac{k - (\ell - 1)m \bmod k}{m} \; , \tag{6}$$

and since $k \le m$ it follows that

$$-1 < -\frac{\ell m \bmod k}{m} \le 0 \; . \tag{7}$$

Substituting (6) on the left and (7) on the right in (5) yields

$$\ell - 1 + \frac{1}{m} \le \frac{ik}{m} < \ell \; ,$$

and hence

$$\left\lceil \frac{ik}{m} \right\rceil = \ell \; . \qquad \square$$

We next show that $U_j\langle f^m\rangle_k$ belongs to the correct set F_r^n, and we locate the "0" terms in its gaps vector.

Proposition 1. *$U_j\langle f^m\rangle_k$ is an element of $F_{jm+k}^{(2j-1)m+2k}$, and its gaps vector has*

$$d_i = \begin{cases} 0 \text{ if } i = jp + \kappa_p, & p \in \{1, \ldots, m\} \\ 1 \text{ otherwise} \end{cases} \tag{8}$$

for $j \in \mathbb{N}$ and $i \in \{1, \ldots, jm+k\}$.

Proof. For $\ell \in \{1, \ldots, k\}$, let f_{i_ℓ} be the active markers in $\langle f^m\rangle_k$. For $j \geq 1$, the insertion of $(gf)^j$ to the left of any f_{i_ℓ} has the effect of replacing e_{i_ℓ} with the $(j+1)$-tuple $B_{j,i_\ell} = (1, 1, \ldots, 1, 0)$. On the other hand, for $j \geq 2$, the insertion of $(gf)^{j-1}$ to the left of any inactive marker f_i, $i \neq i_\ell$, replaces e_i with the j-tuple $B_{j,i} = (1, 1, \ldots, 1, 0)$. Since each of the m terms of $E\langle f^m\rangle_k$ was replaced by a j-tuple, except for the k terms e_{i_ℓ} which were replaced by $(j+1)$-tuples, the total length of the gaps vector (which is to say, the f-rank) of $U_j\langle f^m\rangle_k$ is $jm+k$. And because m terms of this gaps vector are 0, the number of gs in $U_j\langle f^m\rangle_k$ is $jm+k-m = (j-1)m+k$. The density of $U_j\langle f^m\rangle_k$ is therefore

$$\alpha = \frac{(j-1)m+k}{jm+k} \; .$$

This matches the density given in (1), and shows that $U_j\langle f^m\rangle_k$ is an element of $F_{jm+k}^{(2j-1)m+2k}$.

Since $\alpha < 1$ for $j \geq 1$, it follows that, for every $p \in \{1, \ldots, m\}$, the $B_{j,p}$ defined above are in fact Chisala blocks. If an element d_i is the νth element of block $B_{j,p}$, then

$$i = |B_{j,1}| + |B_{j,2}| + \cdots + |B_{j,p-1}| + \nu \; .$$

When $\nu = |B_{j,p}|$ (that is, when d_i is the last element of the pth block), we have

$$\begin{aligned} i &= |B_{j,1}| + |B_{j,2}| + \cdots + |B_{j,p-1}| + |B_{j,p}| \\ &= jp + (\text{sum of the extra 1s in each active block}) \\ &= jp + \kappa_p \; . \end{aligned}$$

Since the only zero element of a block is the last one, Equation (8) follows. □

Proposition 2. *Given $j, k, m \in \mathbb{N}$ with $k \in \{1, \ldots, m\}$, let*

$$\theta(x) = \left\lceil x \frac{(j-1)m+k}{jm+k} \right\rceil, \quad x \in \mathbb{Z}_+ \; . \tag{9}$$

For $p \in \{1,\dots,m\}$ *and* $i \in \{0,\dots,j-1\}$, *let* $X_i = jp + \kappa_p - i$. *Then*

$$\theta(X_i) - \theta(X_i - 1) = \begin{cases} 0 \text{ if } i = 0 \\ 1 \text{ if } i \in \{1,\dots,j-1\} \end{cases} .$$

Some explanation is warranted before we begin the proof. The function θ has been introduced because the code of $U_j\langle f^m\rangle_k$ will be shown to be

$$(\theta(1), \theta(2), \dots, \theta(jm+k)) \ .$$

We have seen that the 0 terms of the gaps vector for $U_j\langle f^m\rangle_k$ occur at the rightmost ends of the Chisala blocks, specifically for index values equal to $jp+\kappa_p$ for $p \in \{1,\dots,m\}$. The expression X_i in Proposition 2 is an index function that allows us to count backward from the 0 term of the pth block as i goes from 1 to $j-1$. (It should be noted that $i \in \{1,\dots,j-1\}$ for p active mod k, and $i \in \{1,\dots,j-2\}$ for p inactive; both cases are covered by the first.)

Proof. Let us more conveniently write

$$\theta(x) = x - \left\lfloor x \frac{m}{jm+k} \right\rfloor ,$$

which follows since $\lceil -N \rceil = -\lfloor N \rfloor$ for any real N. Then, using $\kappa_p = \lceil pk/m \rceil$ from Lemma 1, we have

$$\theta(X_i) = \theta(jp + \kappa_p - i) = jp + \left\lceil \frac{pk}{m} \right\rceil - i - \left\lfloor \left(jp + \left\lceil \frac{pk}{m} \right\rceil - i \right) \frac{m}{jm+k} \right\rfloor .$$

Apply identity (3) to the term $\lceil pk/m \rceil$ inside the floor expression on the right, and simplify:

$$\begin{aligned} \theta(X_i) = jp + \left\lceil \frac{pk}{m} \right\rceil - i - \\ \left\lfloor \frac{jpm}{jm+k} + \left(\frac{pk}{m} + \frac{(-pk) \bmod m}{m} \right) \frac{m}{jm+k} - \frac{im}{jm+k} \right\rfloor \\ = (j-1)p + \left\lceil \frac{pk}{m} \right\rceil - i - \left\lfloor \frac{-im + (-pk) \bmod m}{jm+k} \right\rfloor . \end{aligned} \tag{10}$$

Similarly,

$$\begin{aligned}\theta(X_i - 1) &= jp + \left\lceil \frac{pk}{m} \right\rceil - (i+1) - \left\lfloor \left(jp + \left\lceil \frac{pk}{m} \right\rceil - (i+1) \right) \frac{m}{jm+k} \right\rfloor \\ &= jp + \left\lceil \frac{pk}{m} \right\rceil - (i+1) - \\ &\left\lfloor \frac{jpm}{jm+k} + \left(\frac{pk}{m} + \frac{(-pk) \bmod m}{m} \right) \frac{m}{jm+k} - \frac{(i+1)m}{jm+k} \right\rfloor \\ &= (j-1)p + \left\lceil \frac{pk}{m} \right\rceil - (i+1) - \left\lfloor \frac{-(i+1)m + (-pk) \bmod m}{jm+k} \right\rfloor . \qquad (11)\end{aligned}$$

We now focus on the floor expressions in equations (10) and (11).

Case 1: $i = 0$. When $i = 0$, the floor expression in (10) reduces to

$$\left\lfloor \frac{(-pk) \bmod m}{jm+k} \right\rfloor ,$$

which is identically 0 because $(-pk) \bmod m \leq m-1 < jm+k$ for $k \in \{1, \dots, m\}$ and $j \geq 1$. Thus

$$\theta(X_0) = (j-1)p + \left\lceil \frac{pk}{m} \right\rceil .$$

For Equation (11), when $i = 0$ we have

$$-m \leq -m + (-pk) \bmod m \leq -1 ,$$

and consequently

$$\left\lfloor \frac{-m + (-pk) \bmod m}{jm+k} \right\rfloor = -1 .$$

Therefore

$$\theta(X_0 - 1) = (j-1)p - 1 + \left\lceil \frac{pk}{m} \right\rceil + 1 = \theta(X_0) .$$

Case 2: $i \in \{1, \dots, j-1\}$. Note that this range of i requires $j \geq 2$ for p active mod k, and $j \geq 3$ for p inactive. From

$$0 \leq (-pk) \bmod m \leq m-1$$

and

$$-(j-1)m \leq -im \leq -m$$

we have

$$-1 \leq \frac{-(j-1)m}{jm+k} \leq \frac{-im + (-pk) \bmod m}{jm+k} \leq \frac{-1}{jm+k} < 0 .$$

Hence

$$\left\lfloor \frac{-im + (-pk) \bmod m}{jm + k} \right\rfloor = -1 \ ,$$

and by substitution in (10) it follows that

$$\theta(X_i) = (j-1)p + \left\lceil \frac{pk}{m} \right\rceil - i + 1 \ . \tag{12}$$

A similar calculation shows that

$$-1 \le \frac{-jm}{jm+k} \le \frac{-(i+1)m + (-pk) \bmod m}{jm+k} \le \frac{-(m+1)}{jm+k} < 0 \ ,$$

which implies

$$\left\lfloor \frac{-(i+1)m + (-pk) \bmod m}{jm + k} \right\rfloor = -1 \ . \tag{13}$$

We conclude from (11) and (13) that, for $i \in \{1, \ldots, j-1\}$,

$$\theta(X_i - 1) = (j-1)p + \left\lceil \frac{pk}{m} \right\rceil - i - 1 + 1 = \theta(X_i) - 1 \ ,$$

and the proof is complete. □

Proof of Theorem 1. Proposition 2 implies that, for $x \in \{1, \ldots, jm+k\}$,

$$\theta(x) - \theta(x-1) = \begin{cases} 0 \text{ if } x = jp + \kappa_p, & p \in \{1, \ldots, m\} \\ 1 \text{ otherwise} \ . \end{cases}$$

But Proposition 1 gives the same values at the same indices for the gaps d_x. (Note that $d_0 = 0$, as defined in Definition 1d.) Therefore

$$d_x = \theta(x) - \theta(x-1)$$

for $x \in \{1, \ldots, jm+k\}$, which implies that the code of $U_j\langle f^m \rangle_k$ is

$$(\theta(1), \theta(2), \ldots, \theta(jm+k)) \ .$$

Thus, by the Maximal Perigee Property, $U_j\langle f^m \rangle_k$ is the maximal perigee word in $P^{(2j-1)m+2k}_{jm+k}$. We conclude that $U_j\langle f^m \rangle_k$ is the jth element of $\mathcal{A}^m_k$. □

To close this section, we prove a lemma that gives the code terms corresponding to the 0 terms of the gaps vector for $U_j\langle f^m \rangle_k$.

Lemma 2. *For* $p \in \{1, \ldots, m\}$, *let* $\nu_p = jp + \kappa_p$. *Then the code for* $U_j\langle f^m\rangle_k$ *has* $q_{\nu_p} = (j-1)p + \kappa_p$.

Proof. In the gaps vector for $U_j\langle f^m\rangle_k$, the sum of the elements of Chisala blocks $B_{j,1}, \ldots, B_{j,p}$ is

$$\begin{aligned} q_{\nu_p} &= |B_{j,1}| + \ldots + |B_{j,p}| \\ &= jp + \kappa_p - (\text{number of zero terms in } p \text{ blocks}) \\ &= jp + \kappa_p - p = (j-1)p + \kappa_p \ . \end{aligned} \qquad \square$$

3. Branch functions

The structure of the perigee words in $\mathcal{A}_k^m$ makes it possible to calculate the corresponding perigee values directly, using the parameters a, b, m, and k.

Theorem 2. *The perigees of all words in the* k*th branch of the* m*th maximal family of* $\Psi(a,b)$ *lie on the curve*

$$\Phi_{m,k}(x) = \frac{1}{1-ab} \cdot \frac{1}{1-a^k b^{k-m} x^m} \left[b + \frac{1-b}{a} \sum_{i=1}^{m} \frac{(ab)^{\kappa_i}}{b^i} x^i - \frac{(ab)^{k+1}}{ab^m} x^m \right] . \tag{14}$$

Specifically, for $j \in \mathbb{Z}_+$, *the perigee of* $U_j\langle f^m\rangle_k$ *is* $\Phi_{m,k}((ab)^j)$.

We call $\Phi_{m,k}$ the kth *branch function* for $\mathcal{A}^m$.

Proof. We have seen that the gaps vector for $U_j\langle f^m\rangle_k$ is composed of Chisala blocks, each filled with 1s except for the last term which is 0. Another way to partition the gaps vector is to call the first contiguous group of 1s the first *segment*; subsequent segments begin with a 0 term and contain all 1s up to but not including the next 0 term. This partitioning of the gaps vector translates directly into a partitioning of the code into arithmetic progressions of common difference 1. We apply this observation to the formula (from Proposition 1 in [3]) for the cycle point of an arbitrary $w \in F_r^n$. This gives the perigee of $U_j\langle f^m\rangle_k$ as

$$\frac{1}{1-(ab^{\alpha})^{jm+k}} \sum_{i=1}^{jm+k} b^{q_i} a^{i-1} \ . \tag{15}$$

Set $x = (ab)^j$. The density of $U_j\langle f^m\rangle_k$ is

$$\alpha = \frac{(j-1)m+k}{jm+k} \ ,$$

and hence

$$\frac{1}{1-(ab^{\alpha})^{jm+k}} = \frac{1}{1-a^k b^{k-m} x^m} \; . \tag{16}$$

We now break the summation in (15) into sums over segments of the code for $U_j\langle f^m\rangle_k$. Of these $m+1$ sums, the last comprises a single term, while the remaining m can be written as geometric series from which have been factored terms involving x, a, b, and the κ_i. Abusing the summation notation slightly, we begin with

$$\sum_{i=1}^{jm+k} b^{q_i} a^{i-1} = \left(\sum_{i=1}^{j+\kappa_1-1} + \sum_{i=j+\kappa_1}^{2j+\kappa_2-1} + \cdots + \sum_{\substack{i= \\ j(m-1)+\kappa_{m-1}}}^{jm+\kappa_m-1} \right) (b^{q_i} a^{i-1}) + \frac{(ab)^k}{ab^m} x^m \; , \tag{17}$$

where by Lemma 2 the last term is a rearrangement of

$$b^{q_{jm+k}} a^{jm+k-1} = b^{(j-1)m+k} a^{jm+k-1} \; . \tag{18}$$

Of the remaining m summations over segments in (17), consider the first. Invariably $\kappa_1 = 1$, so $j + \kappa_1 - 1 = j$. The sum of a finite geometric series and the substitution $x = (ab)^j$ then yield

$$\sum_{i=1}^{j} b^{q_i} a^{i-1} = b \sum_{i=0}^{j-1} (ab)^i = b \frac{1-x}{1-ab} \; . \tag{19}$$

Of the remaining $m-1$ sums, consider the pth, $p \in \{1, \ldots, m-1\}$. Its first summand—corresponding to the segment's 0 term—is

$$b^{(j-1)p+\kappa_p} a^{jp+\kappa_p-1} = \frac{(ab)^{\kappa_p}}{ab^p} x^p \; , \tag{20}$$

where the exponent of b on the left follows from Lemma 2. This term can be factored out of the entire pth sum, which then becomes

$$\begin{aligned} \sum_{i=jp+\kappa_p}^{j(p+1)+\kappa_{p+1}-1} b^{q_i} a^{i-1} &= \frac{(ab)^{\kappa_p} x^p}{ab^p} \sum_{i=0}^{j+e_{p+1}-1} (ab)^i \\ &= \frac{(ab)^{\kappa_p} x^p}{ab^p} \cdot \frac{1-x(ab)^{e_{p+1}}}{1-ab} \\ &= \frac{(ab)^{\kappa_p} x^p - (ab)^{\kappa_{p+1}} x^{p+1}}{ab^p(1-ab)} \; . \end{aligned} \tag{21}$$

In this calculation we used the relationship between the activation vector and the activation counters given by $\kappa_i + e_{i+1} = \kappa_{i+1}$ for $i \in \{1, \ldots, m-1\}$.

Finally, sum (21) over all $p \in \{1, \ldots, m-1\}$, and add the first term obtained in (19). Equation (14) is obtained by factoring out the divisor $1/(1-ab)$ and gathering like powers of x. □

While substitution of specific rational numbers for a and b in formula (14) leads to some simplification, there are also two special cases in which the general form (14) can be compressed. First, we prove the interesting fact that when $k = m$, the branch function is a degree 1 rational function independent of m.

Corollary 1. *The mth branch function for $\mathcal{A}^m$ is*

$$\Phi_{m,m}(x) = \frac{1}{1-ab}\left(b + \frac{(1-b)x}{1-ax}\right) . \tag{22}$$

PROOF SKETCH. Every $i \in \{1, \ldots, m\}$ is active mod m. Thus every element e_i of the activation vector for $\langle f^m \rangle_m$ is equal to 1, and hence $\kappa_i = i$ for all $i \in \{1, \ldots, m\}$. This and the substitution $k = m$ reduce (14) to

$$\Phi_{m,m}(x) = \frac{1}{1-ab} \cdot \frac{1}{1-(ax)^m}\left[b + \frac{1-b}{a}\sum_{i=1}^{m}(ax)^i - b(ax)^m\right] .$$

Summing the geometric series and simplifying yields (22). □

Corollary 2. *If p is a common factor of k and m, so that $k = pk'$ and $m = pm'$, then $\Phi_{m,k}(x) = \Phi_{m',k'}(x)$.*

PROOF SKETCH. At the heart of this result is the fact that $\langle f^m \rangle_k$ consists of p copies of $\langle f^{m'} \rangle_{k'}$. Details are left to the reader. □

As an example illustrating Theorem 2 and its corollaries, Table 4 shows the branch functions, in lowest terms, for $\mathcal{A}^1$, $\mathcal{A}^2$, $\mathcal{A}^3$, and $\mathcal{A}^4$ in $\Psi(\frac{3}{2}, \frac{1}{2})$.

Such functions might prove useful in problems concerning the existence of integer-valued perigees. In the example above, for instance, one can show that $\Phi_{m,m}(x) = 2(2-x)/(2-3x)$ attains only two integer values, -10 and -2, when x is a nonnegative power of $ab = 3/4$.

References

[1] B. P. Chisala, Cycles in Collatz sequences, *Publ. Math. Debrecen* **45** (1994), 35–39.

Table 4. The branch functions $\Phi_{m,k}$ for $m \in \{1, \ldots, 4\}$ in $\Psi(\frac{3}{2}, \frac{1}{2})$.

$k \backslash m$	1	2	3	4
1	$\frac{2(2-x)}{2-3x}$	$\frac{2(1+x-x^2)}{1-3x^2}$	$\frac{2(1+x+2x^2-2x^3)}{1-6x^3}$	$\frac{2(1+x+2x^2+4x^3-4x^4)}{1-12x^4}$
2		$\frac{2(2-x)}{2-3x}$	$\frac{2(2+2x+3x^2-3x^3)}{2-9x^3}$	$\frac{2(1+x-x^2)}{1-3x^2}$
3			$\frac{2(2-x)}{2-3x}$	$\frac{2(4+4x+6x^2+9x^3-9x^4)}{4-27x^4}$
4				$\frac{2(2-x)}{2-3x}$

[2] R. L. Graham, D. E. Knuth, O. Patashnik, Concrete Mathematics, 2nd ed., *Addison-Wesley Publishing Co.*, *Reading, MA*, 1994.

[3] D. J. Jones, Parameter-independent structure in periodic orbits of an iterated function system on the real line, *Publ. Math. Debrecen* (to appear).

5112 FAIRCHILD AVE
FAIRBANKS, ALASKA USA 99709-4523

E-mail: djjones@alaska.edu