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QUADRANGLES, BUTTERFLIES, PASCAL'S HEXAGON, AND PROJECTIVE FIXED POINTS

DIXON JONES

It is a consequence of an important result in projective geometry, the so-called Fundamental Theorem on Quadrangular Sets, that if five sides of a complete quadrangle pass through fixed points on a given line, then the sixth side also meets that line at a fixed point; further, it may be shown that this theorem is equivalent to Desargues' Theorem for perspective triangles. In this article we exhibit results of a similar flavor but involving only simple quadrangles and the Theorem of Pascal and carrying implications for the classical Butterfly Theorem and one of its recent extensions.

(We employ the notion of lines of a configuration which pass through "fixed points" on a given line and, dually, points of a configuration which lie on "fixed lines" through a given point. This is used as a shorthand for more rigorous but less economical statements about configurations with sides and vertices placed in one-to-one correspondence such that corresponding sides meet at points of a given line, and dually; thus a phrase such as "the sides of a quadrangle meet a given line at fixed points" should be understood to mean, equivalently, "the sides of a second quadrangle, distinct from the first, whose vertices have been placed in one-to-one correspondence with those of the first meet the corresponding sides of the first at points on the given line." The "fixedness" of these points can be made more tangible by imagining that the quadrangle moves about continuously, as in the real projective plane, but that its sides always pass through collinear points which do not move.)

Figure 1 illustrates our first and primary result.

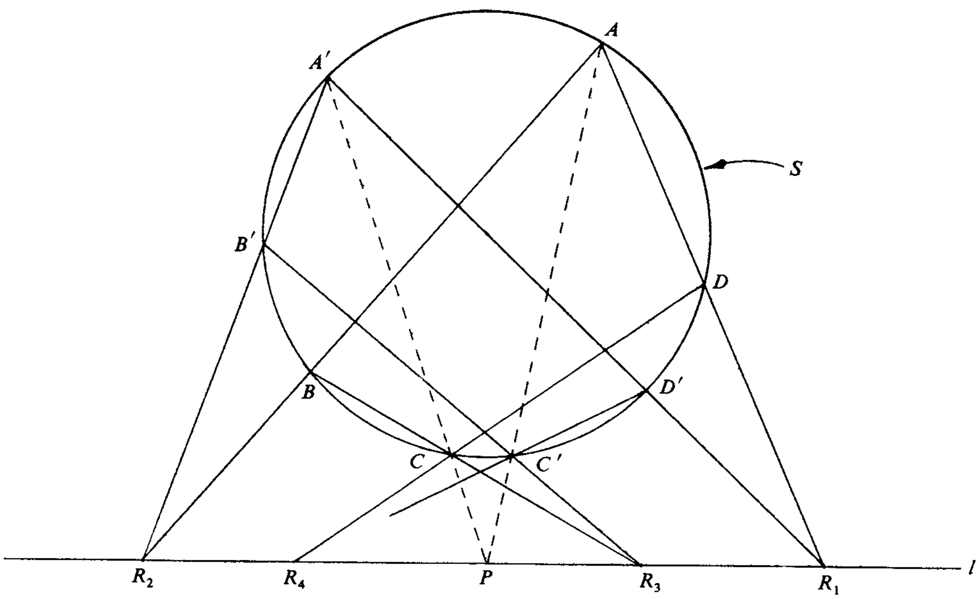


FIG. 1

THEOREM 1. *If three sides of a simple quadrangle inscribed in a point conic pass through fixed points on a given line, then the fourth side also passes through a fixed point on that line. Dually, if three vertices of a simple quadrilateral escribed about a line conic lie on fixed lines through a given point, then the fourth vertex also lies on a fixed line through that point.*

Proof. Referring to Figure 1, let quadrangles $ABCD$ and $A'B'C'D'$ be inscribed in a conic S, with three pairs of corresponding sides intersecting at points R_1, R_2, and R_3 of a given line l. CD meets l at R_4; it must be shown that $C'D'$ also meets l at R_4. Consider the hexagon $ABCA'B'C'$. By the Theorem of Pascal, since opposite sides AB and $A'B'$ meet on l at R_2, and BC and $B'C'$ meet on l at R_3, it follows that $A'C$ and AC' also meet on l, say at P. Now consider hexagon $ADCA'D'C'$. Again by Pascal's Theorem, since opposite sides AD and $A'D'$ meet on l at R_1, and $A'C$ and AC' meet on l at P, it follows that CD and $C'D'$ meet on l, in fact at the point R_4, which completes the proof. The dual follows by the principle of duality in the plane.

Theorem 1 represents the simplest case of the following more general theorem.

THEOREM 2. *Let an n-gon, $n=2k$, be inscribed in a point conic and let $n-1$ of its sides meet a given line at fixed points. Then the nth side also meets that line at a fixed point. Dually, let an n-lateral, $n=2k$, be escribed about a line conic and let $n-1$ of its vertices lie on fixed lines through a given point. Then the nth vertex also lies on a fixed line through that point.*

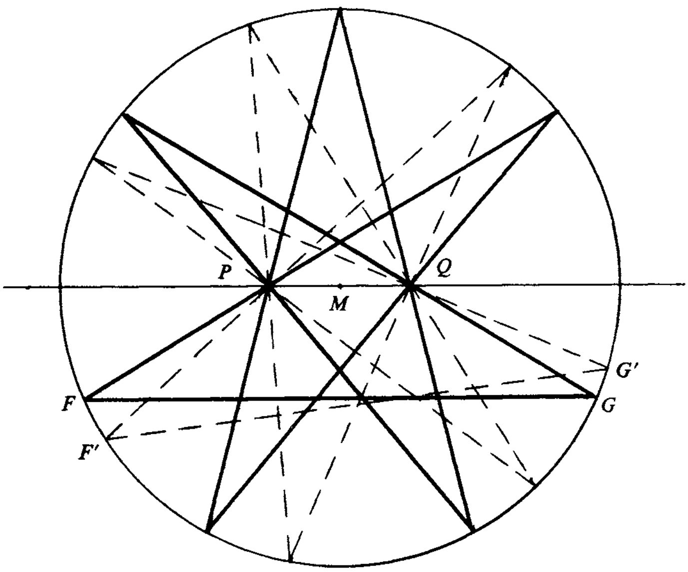
Proof. Our proof will be by induction on k. Clearly, the simplest polygon of $2k$ vertices occurs when $k=2$, and in this initial case Theorem 2 reduces to Theorem 1, which has been proved. Suppose now that Theorem 2 holds for some integer k; it must be shown to hold for $k+1$. Let an inscribed polygon P have consecutive vertices $P_1, P_2, \ldots, P_n, P_{m-1}, P_m$, where $m=2(k+1)$. It is given that $m-1$ sides of P meet a given line l at fixed points; without loss of generality, let P_1P_2 through $P_{m-1}P_m$ be the sides so given. By supposition, the theorem holds for a polygon of $n=2k$ sides; specifically, since the sides P_1P_2 through $P_{n-1}P_n$ of the polygon $P_1P_2 \cdots P_n$ pass through fixed points on l, the nth side P_1P_n also meets l at a fixed point. Now, P_1P_n, P_nP_{m-1}, $P_{m-1}P_m$, and P_mP_1 are the sides of an inscribed quadrangle, of which sides three are given or have been shown to meet l at fixed points; hence P_mP_1, the fourth side of the quadrangle and the mth side of P, also meets l at a fixed point, by Theorem 1. We have thus shown, by assuming its validity for $n=2k$, that Theorem 2 holds for $m=2(k+1)$, which completes the proof. The dual, of course, follows automatically.

One would naturally wonder if Theorem 2 holds for odd n. Unfortunately, the following construction disposes of any hopeful speculation on the matter. Given a circle, a chord therein, and points P and Q on the chord equidistant from its midpoint, construct for n an odd integer an inscribed n-gon with one vertex on the chord's perpendicular bisector, such that $n-1$ sides alternately pass through P and Q. Due to the bilateral symmetry imposed on the n-gon in this way, the nth side will be parallel to PQ. By perturbing any vertex to yield a second n-gon also passing alternately through P and Q $n-1$ times, it will be seen that the nth side cannot be parallel to PQ. Thus, while the two n-gons meet the conditions of Theorem 2, the result does not follow. Figure 2 illustrates this construction for the case $n=7$.

We mentioned in an earlier comment that the Fundamental Theorem on Quadrangular Sets is equivalent to Desargues' Theorem for perspective triangles. An equally intimate relationship is shared by Theorem 1 and another celebrated projective theorem.

THEOREM 3. *Theorem 1 is equivalent to the Theorem of Pascal.*

Proof. It was shown in our first proof that Pascal's Theorem implies Theorem 1; the reverse implication remains to be shown. Let hexagon $ABCDEF$ be inscribed in a conic, let AB meet DE at P, BC meet EF at Q, CD meet PQ at R, and let AD meet PQ at T. We discern that the



FG is parallel to *PQ*, but *F'G'* is not.

FIG. 2

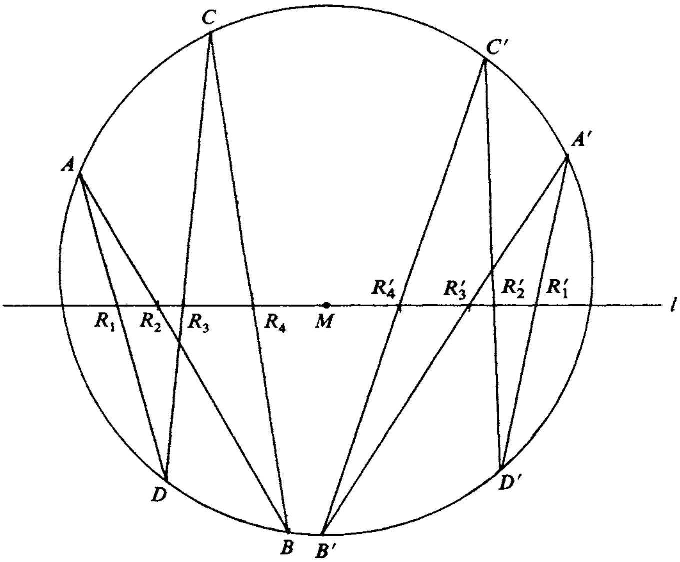


FIG. 3

quadrangles *ABCD* and *DEFA* possess three pairs of corresponding sides, those which pass through the points *P*, *Q*, and *T* on *PQ*, and hence the fourth pair of corresponding sides, *CD* and *AF*, meet *PQ* at the point *R*, which completes the proof.

Our final result is presented as a matter of incidental interest, to indicate a further connection between Pascal's Theorem and the notion of fixed points on a line.

THEOREM 4. *If five sides of a hexagon inscribed in a point conic meet a given line at fixed points, then its Pascal line meets that line at a fixed point.*

Sketch of proof. With reference to the construction and notation used in the previous proof, let AB, CD, DE, EF, and AF be the sides meeting a given line l at fixed points. Application of Theorem 1 to the simple quadrangle $ADEF$ yields that AD meets l at a fixed point, and application of the Fundamental Theorem on Quadrangular Sets to the complete quadrangle $PADR$ yields that PR, the Pascal line of $ABCDEF$, meets l at a fixed point. Details of the proof are left to the reader.

As a concluding remark, we observe that Theorem 1 is a generalization of a recent result called the Double Butterfly Theorem [1], which states that if two "butterflies" (re-entrant quadrangles $ABCD$ and $A'B'C'D'$) inscribed in a circle have their respective sides meeting a chord of the circle at R_1, R_2, R_3, R_4 and R_1', R_2', R_3', R_4', and if the chord's midpoint M bisects the segments R_1R_1', R_2R_2', and R_3R_3', then M bisects R_4R_4' (Fig. 3). Theorem 1 generalizes this by removing the midpoint and its metric aspects, by allowing all simple quadrangles, by removing the restriction that the given line intersect the circle, and by including conics other than the circle. Indeed, the proof of the Double Butterfly Theorem becomes a simple matter using Theorem 1: referring to Figure 3, map $ABCD$ to $A''B''C''D''$ by reflection in the midpoint M. By hypothesis, therefore, three corresponding sides of $A'B'C'D'$ and $A''B''C''D''$ pass through R_1', R_2', and R_3'. By Theorem 1, the fourth sides $B'C'$ and $B''C''$ pass through R_4', and since $B''C''$ is a reflection of BC in M, we have $R_4M = R_4'M$. That the Double Butterfly Theorem implies the classical Butterfly Theorem was pointed out in [1]; to prove the latter result directly using Theorem 1, align $ABCD$ and $A'B'C'D'$ so that $R_1 = R_4'$ and $R_2 = R_2' = R_3 = R_3' = M$, assume that R_4 is distinct from R_1', and apply the above argument.

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THE RETRIAL OF THE LOWER SLOBBOVIAN COUNTERFEITERS

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Lower Slobbovia is a poor country, unable to mint its local currency, the Rasbucknik [1]. Hence nine (N) coiners, $C_1, C_2, \ldots, C_9$ were engaged to produce coins to government specifications. However, it was suspected that some of the coiners were counterfeiting by introducing some base metal into the alloy. Any pair of counterfeit coins weighed the same, but differed slightly in weight from good coins. Each coiner produced either all good coins or all counterfeits. A procedure was called for to determine in three weighings which (if any) of the coiners were dishonest, using a beam balance with a set of infinitely refinable weights, as many coins from each coiner as may be needed, and one good coin.

The Court of Lower Slobbovia was supplied with the following solution [2]. In the first weighing, the weight Wg of the good coin is determined. In the second weighing, a single coin from each coiner is selected and these nine (N) coins are weighed, giving a total weight T. If the discrepancy $D = T - NWg$ is zero then all the coiners are honest. In the third weighing a sample of 2^{i-1} coins are selected from C_i, $i = 1, \ldots, 9$, and weighed, giving a total weight T'. The discrepancy here is $D' = T' - (2^N - 1)Wg$.

Now the integer S such that $D'/D = S/\beta(S)$ is determined where $\beta(S)$ is the number of ones in the binary representation of S, i.e., $\beta(S) = \sum_{i=1}^{N} B_i$ where $S = \sum_{i=1}^{N} B_i 2^{i-1}$. For such an S, coiner C_i is a counterfeiter if and only if $B_i = 1$, and $\beta(S)$ is the number of counterfeiters.

This procedure was followed at the trial of the coiners in 1954 and the fateful ratio $D'/D = 23$ was determined. After further computation it was discovered that $S = 69 = 2^0 + 2^2 +$