Perigees in a real-valued extension of the $3x + 1$ problem

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The $3x + 1$ problem

AKA

Collatz Problem

Syracuse Problem

Kakutani's Problem

Hasse's Algorithm

Ulam's Problem

Hailstone Numbers

Wondrous Numbers

Collatz function

$$C: \mathbf{Z} \to \mathbf{Z}$$

$$C(n) = \begin{cases} 3n + 1, & n \equiv 1 \pmod{2} \\ \dfrac{n}{2}, & n \equiv 0 \pmod{2} \end{cases}$$

$3n + 1$ is always even, so…

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$3x + 1$ function

$$T\colon \mathbf{Z} \to \mathbf{Z}$$

$$T_1 = f(x)$$

$$T(x) = \begin{cases} \dfrac{3x+1}{2}, & x \equiv 1 \pmod 2 \\ \dfrac{x}{2}, & x \equiv 0 \pmod 2 \end{cases}$$

$$T_0 = g(x)$$

The $3x + 1$ Conjecture

For every $N_0 \in \mathbf{N}$, there is a $k \in \mathbf{N}$ such that

$$T^k(N_0) = \underbrace{T \circ T \circ \cdots \circ T}_{k \text{ terms}}(N_0) = 1$$

E.g.:

$$N_0 = 5 \xrightarrow[f]{} 8 \xrightarrow[g]{} 4 \xrightarrow[g]{} 2 \xrightarrow[g]{} 1$$

$$N_0 = 7 \xrightarrow[f]{} 11 \xrightarrow[f]{} 17 \xrightarrow[f]{} 26 \xrightarrow[g]{} 13 \xrightarrow[f]{} 20 \xrightarrow[g]{} 10 \xrightarrow[g]{} 5 \rightarrow \ldots$$

Erdös: "Mathematics is not yet ready for such problems."

Known integer cycles

$gf(1) = 1$	$1 \to 2 \to 1.$
$g(0) = 0$	$0 \to 0.$
$f(-1) = -1$	$-1 \to -1.$
$gff(-5) = -5$	$-5 \to -7 \to -10 \to -5.$
$gggfffgffff(-17) = -17$	$-17 \to -25 \to -37 \to -55 \to$ $-82 \to -41 \to -61 \to -91 \to$ $-136 \to -68 \to -34 \to -17.$

The cycles problem seems “easier” than the iteration problem.

A one-dimensional iterated function system*

$$f(x) = ax + b$$
$$g(x) = cx$$

where $a, b \in (0, \infty)$, $c \in (0, 1)$, and $x \in \mathbf{R}.$

*Michael Barnsley, *Fractals Everywhere*, 1988

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Q: How do we define the "conditions" of $T(x)$?

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Q: How do we define the "conditions" of $T(x)$?

A: We don't.

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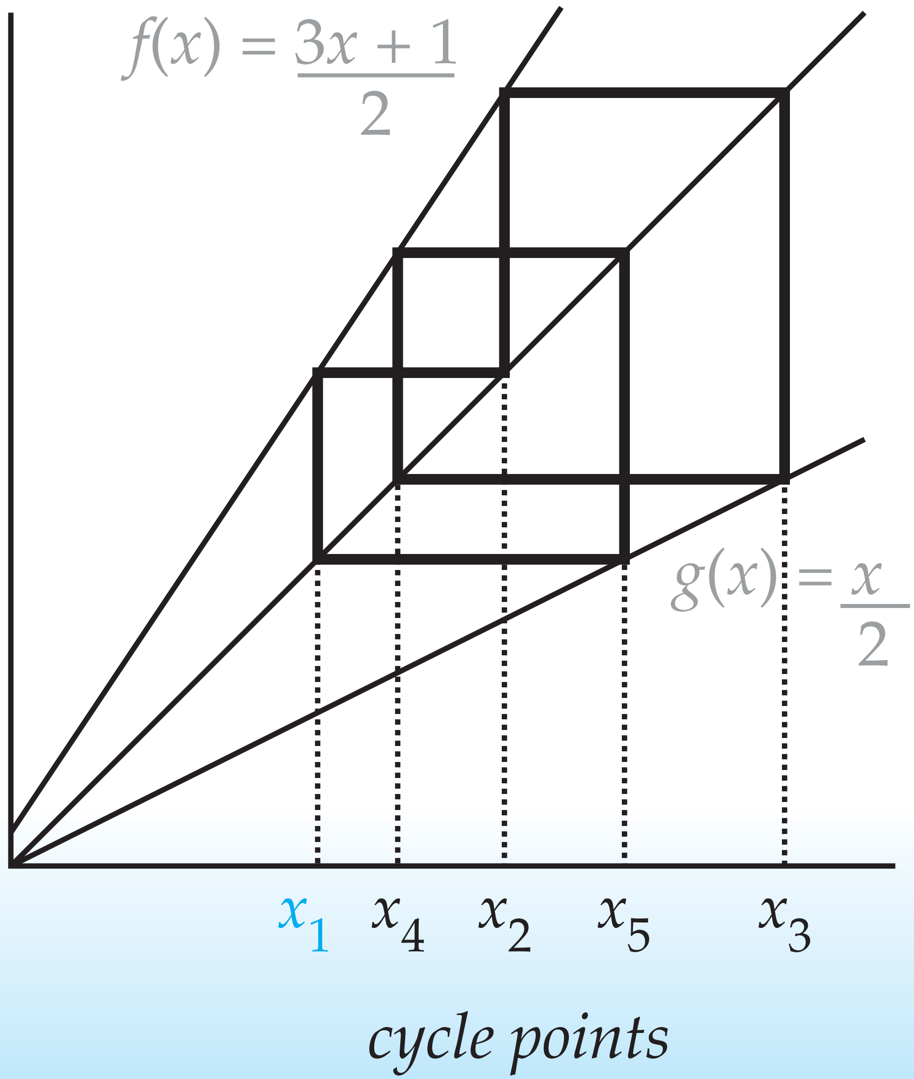
Cycles

Let $t_i \in \{f, g\}$, $i = 1, \ldots, n$. Then

$$t_n t_{n-1} \ldots t_1 (x_1) = x_1$$

defines an n-cycle point x_1.

A typical cycle



perigee

$$gfgff(x_1) = x_1 = 23/5$$

$$fgfgf(x_2) = x_2 = 37/5$$

$$ffgfg(x_3) = x_3 = 58/5$$

$$gffgf(x_4) = x_4 = 29/5$$

$$fgffg(x_5) = x_5 = 46/5$$

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Let $t_i \in \{f, g\}, i = 1, \ldots, n$. Then

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- x_1 always exists (could be infinite), and is unique, because f & g are linear and $c > 0, a, b < \infty$.
- We can view $t_n t_{n-1} \ldots t_1$ as an element in the set Σ^n of n-letter words on 2 symbols.

ggggggggg gfggfgggg ggfggfgfg fggggfggf gggffgfgf gfgfgggff fggfgggff ffgfggggf gfggfffgf fgggfffgf ffgggfffg
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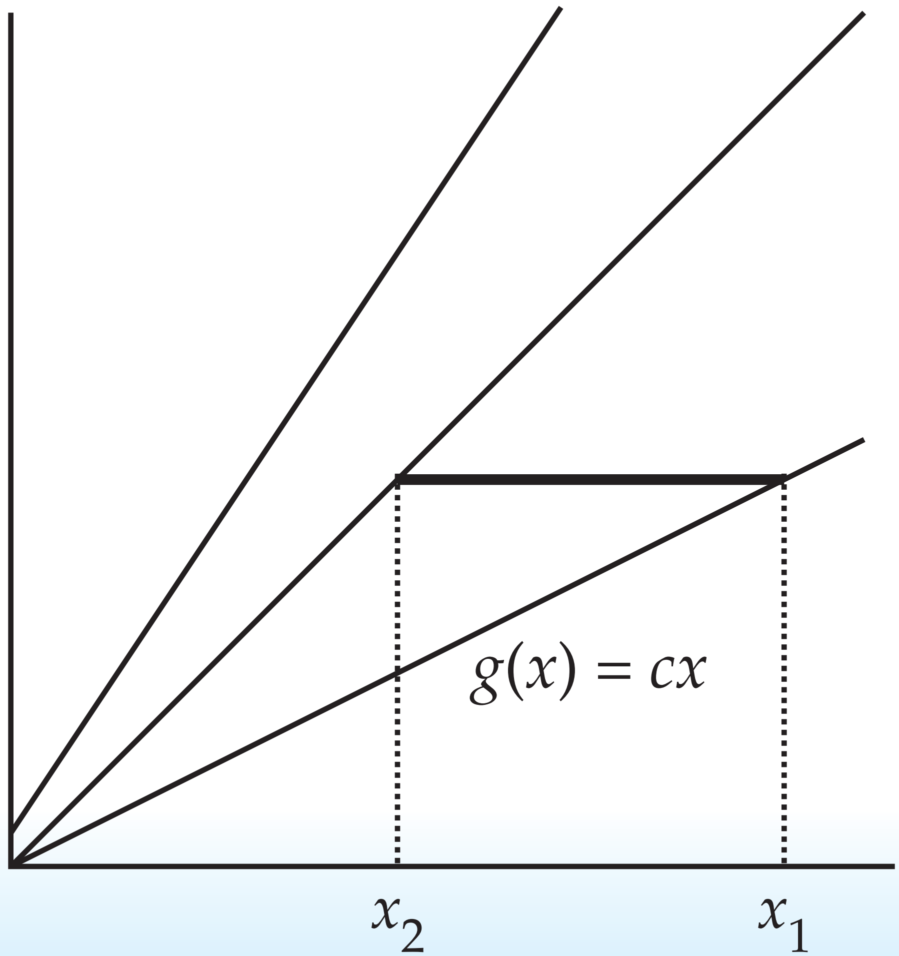
Culling Σ^n

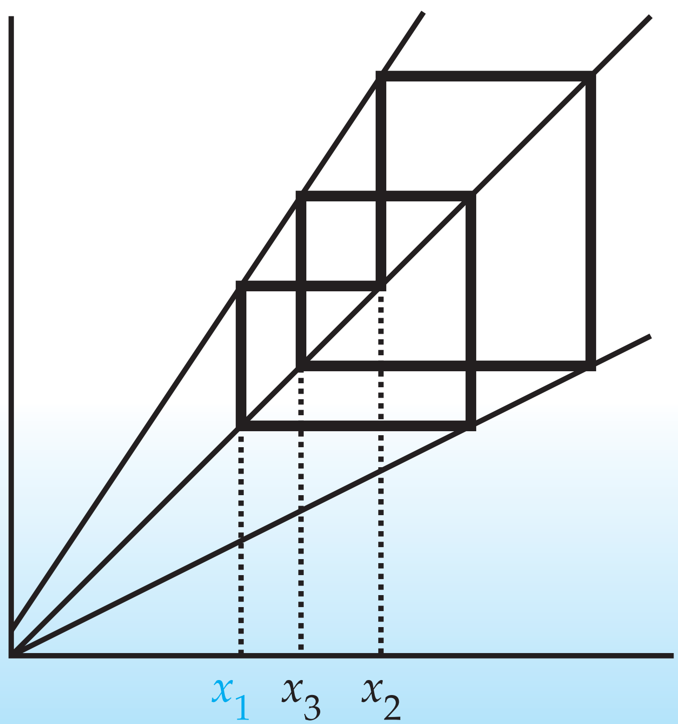


No perigee word
can have g rightmost.

Where the perigees are

$F^n = \{w \in \Sigma^n \mid w \text{ has } f \text{ rightmost}\}$

$F^n_r = \{w \in F^n \mid w \text{ has } r\ f\text{s}\}$



Previous example is reduced to

$gfgff(x_1) = x_1 = 23/5$

$fgfgf(x_2) = x_2 = 37/5$

$gffgf(x_3) = x_3 = 29/5$

Cycle canonical representation

- Each cycle in F_r^n can be represented by one of its cycle words. So, why not choose the perigee word?

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- Each cycle in F_r^n can be represented by one of its cycle words. So, why not choose the perigee word?
- The perigee word is unique, although it may have multiplicity greater than one.
- The ***layer*** $L_{n,r}$ is the set of perigee words in F_r^n.

layer	word	code	perigee
$L_{8,0}$	*gggggggg*	Λ	0.0
$L_{8,1}$	*gggggggf*	(7)	0.00395
$L_{8,2}$	*ggggggff*	(6, 6)	0.02024
	gggggfgf	(5, 6)	0.02834
	ggggfggf	(4, 6)	0.04453
	gggfgggf	(3, 6)	0.07692
$L_{8,3}$	*gggggfff*	(5, 5, 5)	0.08297
	ggggfgff	(4, 5, 5)	0.10044
	ggggffgf	(4, 4, 5)	0.12664
	gggfggff	(3, 5, 5)	0.13537
	gggfgfgf	(3, 4, 5)	0.16157
	ggfgggff	(2, 5, 5)	0.20524
	ggfggfgf	(2, 4, 5)	0.23144
$L_{8,4}$	*ggggffff*	(4, 4, 4, 4)	0.37143
	gggfgfff	(3, 4, 4, 4)	0.41714
	gggffgff	(3, 3, 4, 4)	0.48571
	ggfggfff	(2, 4, 4, 4)	0.50857
	ggfgfgff	(2, 3, 4, 4)	0.57714
	gggfffgf	(3, 3, 3, 4)	0.58857
	ggfgffgf	(2, 3, 3, 4)	0.68
	ggffggff	(2, 2, 4, 4)	0.71429
	ggffgfgf	(2, 2, 3, 4)	0.81714
	gfgfgfgf	(1, 2, 3, 4)	1.

layer	word	code	perigee
$L_{8,5}$	*gggfffff*	(3, 3, 3, 3, 3)	16.23077
	ggfgffff	(2, 3, 3, 3, 3)	17.46154
	ggffgfff	(2, 2, 3, 3, 3)	19.30769
	gfggffff	(1, 3, 3, 3, 3)	19.92308
	gfgfgfff	(1, 2, 3, 3, 3)	21.76923
	ggfffgff	(2, 2, 2, 3, 3)	22.07692
	gfgffgff	(1, 2, 2, 3, 3)	24.53846
$L_{8,6}$	*ggffffff*	(2, 2, 2, 2, 2, 2)	–1.40592
	gfgfffff	(1, 2, 2, 2, 2, 2)	–1.47357
	gffgffff	(1, 1, 2, 2, 2, 2)	–1.57505
	gfffgfff	(1, 1, 1, 2, 2, 2)	–1.72727
$L_{8,7}$	*gfffffff*	(1, 1, 1, 1, 1, 1, 1)	–1.06629
$L_{8,8}$	*ffffffff*	(0, 0, 0, 0, 0, 0, 0, 0)	–1.

Distinct cycles and perigees for $n = 8$ using $T(x)$

Anatomy of a word

$$w = ggffgfggfffgf = ggf_1f_2gf_3ggf_4f_5f_6gf_7$$

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- **gaps vector**. $D(w) = (d_1, d_2, \ldots, d_r)$, where

  d_i = number of gs between f_{i-1} and f_i, indices mod r.

  E.g. $D(w) = (2, 0, 1, 2, 0, 0, 1)$

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- **gaps vector**. $D(w) = (d_1, d_2, \ldots, d_r)$, where
  d_i = number of gs between f_{i-1} and f_i, indices mod r.
  E.g. $D(w) = (2, 0, 1, 2, 0, 0, 1)$
- **code.** $Q(w) = (q_1, q_2, \ldots, q_r)$, $q_i = d_1 + d_2 + \ldots + d_i$.
  E.g. $Q(w) = (2, 2, 3, 5, 5, 5, 6)$

The code yields the cycle point

Proposition. Let $w \in F_r^n$ have density α and code $Q(w) = (q_1, q_2, \ldots, q_r)$, $1 \le r \le n$. Then

$$x_1 = \frac{b}{1 - (ac^{\alpha})^r} \sum_{i=1}^{r} c^{q_i} a^{i-1}$$

is the unique point in **R** satisfying $w(x_1) = x_1$.

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is the unique point in $\mathbf{R}$ satisfying $w(x_1) = x_1$.

Proof sketch: Show by induction on n that

$$w(x) = a^r c^{n-r} x + b \sum_{i=1}^{r} c^{q_i} a^{i-1}.$$

Minimal Perigee Property

The minimal perigee word with density α is $w_{\min} = g^{r\alpha} f^{r}$, whose code is the r-tuple

$$Q(w) = (r\alpha, r\alpha, \ldots, r\alpha).$$

Proof sketch: Easy using the proposition.

Maximal Perigee Property

The maximal perigee word w_{max} with density α has code

$$Q(w_{\text{max}}) = (\lceil\alpha\rceil, \lceil 2\alpha\rceil, \lceil 3\alpha\rceil, \ldots, \lceil r\alpha\rceil)$$

where $\lceil\cdot\rceil$ is the ceiling function.

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$$n = 8, r = 5, \alpha = 3/5$$

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Keeping the density fixed,

- determine the perigee word of an arbitrary n-cycle;

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Keeping the density fixed,

- determine the perigee word of an arbitrary n-cycle;
- find the maximal perigee word for a special value of a;
- show that this perigee word is maximal for all positive real a.

Chisala's Lemma

Any finite sequence of real numbers with average α can be cyclically permuted so that its partial averages are all bounded above by α (or all bounded below by α).

Busiso P. Chisala, Cycles in Collatz sequences, *Publ. Math. Debrecen* **45** (1994), 35–39.

Chisala's Lemma example

$$D = (1, 2, 0, 0, 1, 3, 0), \alpha = 1$$

$$S = (1, \frac{3}{2}, \frac{3}{3}, \frac{3}{4}, \frac{4}{5}, \frac{7}{6}, \frac{7}{7})$$

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$D_\sigma = (0, 1, 2, 0, 0, 1, 3)$

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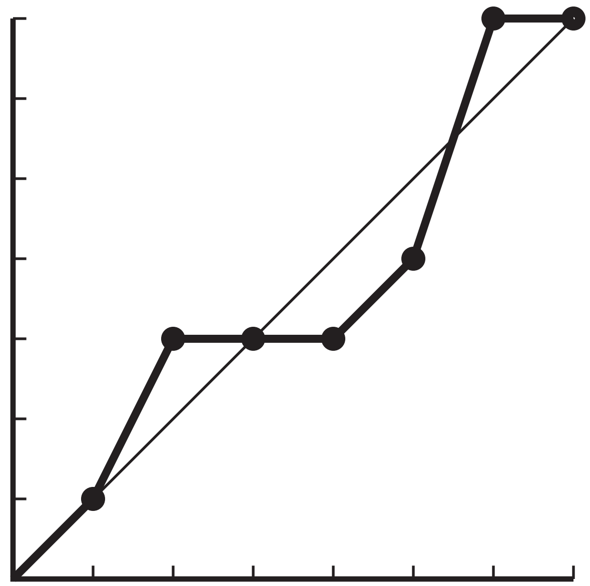
$D_\tau = (3, 0, 1, 2, 0, 0, 1)$

$S_\tau = (3, \frac{3}{2}, \frac{4}{3}, \frac{6}{4}, \frac{6}{5}, \frac{6}{6}, \frac{7}{7})$ all ≥ 1

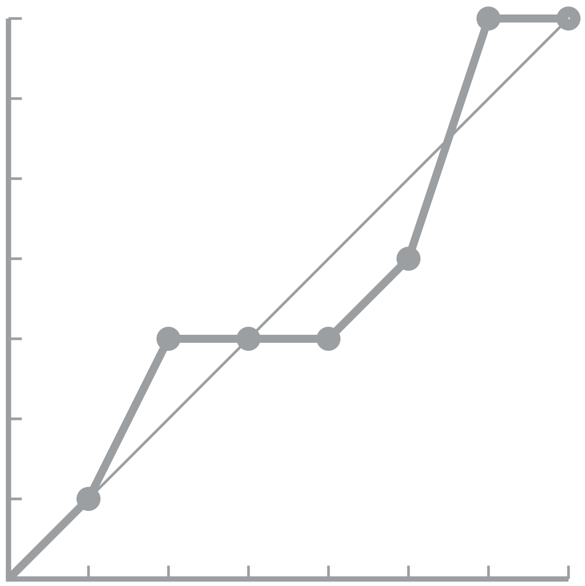
Chisala’s Lemma—another view



$D = (1, 2, 0, 0, 1, 3, 0)$

$Q = (1, 3, 3, 3, 4, 7, 7)$

Chisala's Lemma—another view



$D = (1, 2, 0, 0, 1, 3, 0)$

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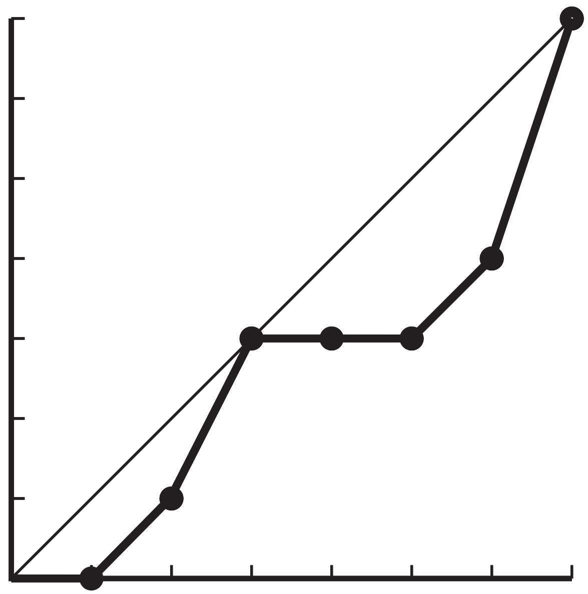

$D_\sigma = (0, 1, 2, 0, 0, 1, 3)$

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A ***subdiagonal*** path

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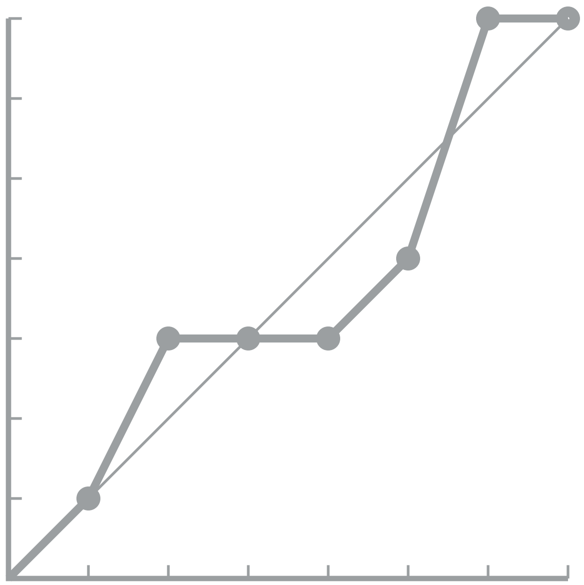

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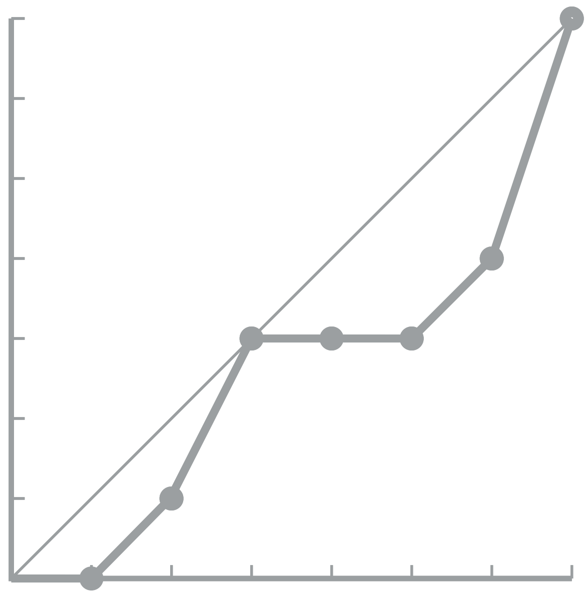

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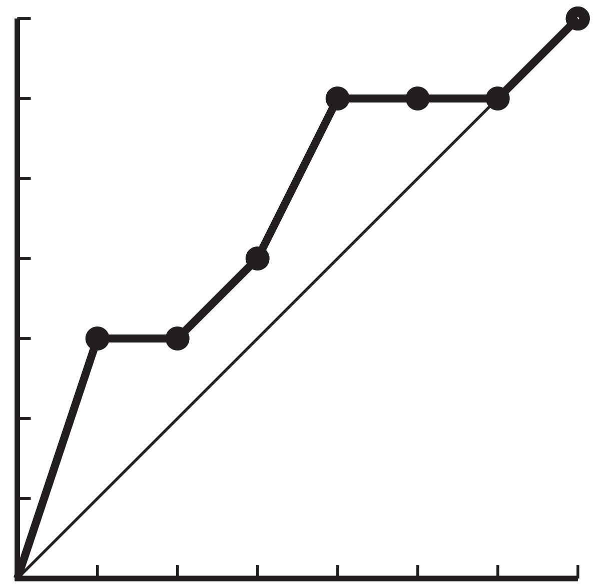

$D_\tau = (3, 0, 1, 2, 0, 0, 1)$

$Q_\sigma = (3, 3, 4, 6, 6, 6, 7)$

A ***superdiagonal*** path

Note: First d_i can't be the minimum.

Some notation

Let $w \in F_r^n$ have density α and gaps vector

$D(w) = (d_1, d_2, \ldots, d_r)$, $1 \leq r \leq n$.

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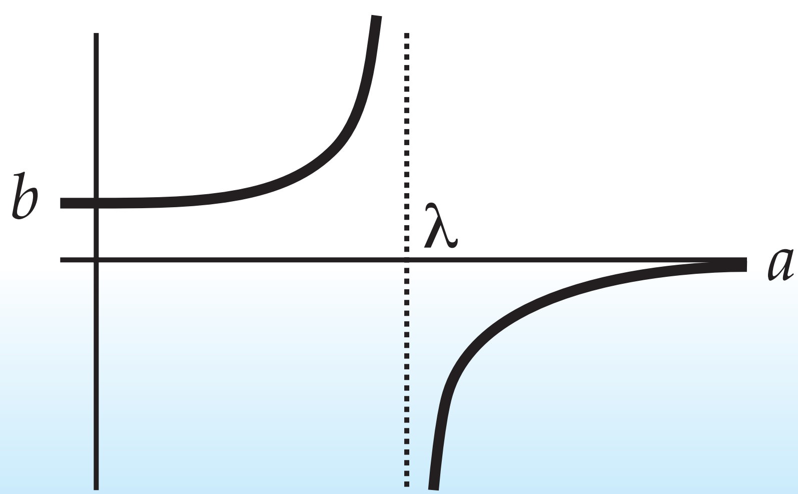
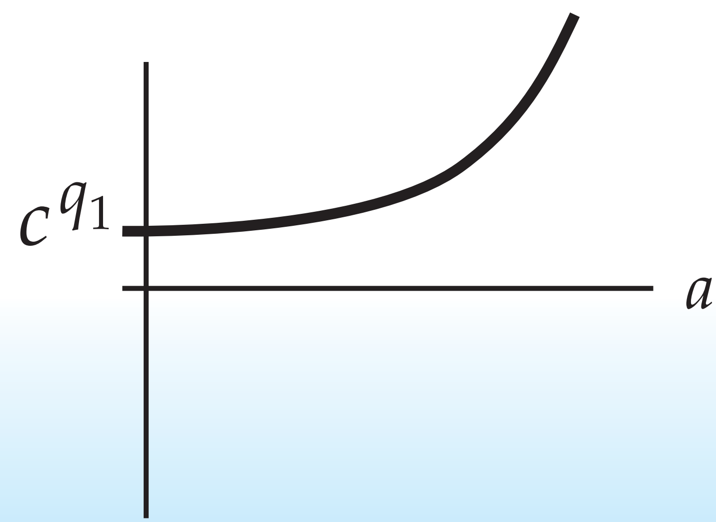
- $\lambda = c^{-\alpha}$.

Code functions

$$x_1 = \underbrace{\frac{b}{1-(ac^{\alpha})^r}}_{v(a)} \underbrace{\sum_{i=1}^{r} c^{q_i} a^{i-1}}_{u_1(a) = u(Q_1,\, a)} \leftarrow \text{first code function}$$



$v(a)$



$u_1(a)$

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- For any rotation w_j of w_1 and any $a \in [0, \lambda)$,

$$0 < x_1 = v(a)u_1(a) < v(a)u_j(a) = x_j \quad \text{iff} \quad u_1(a) < u_j(a).$$

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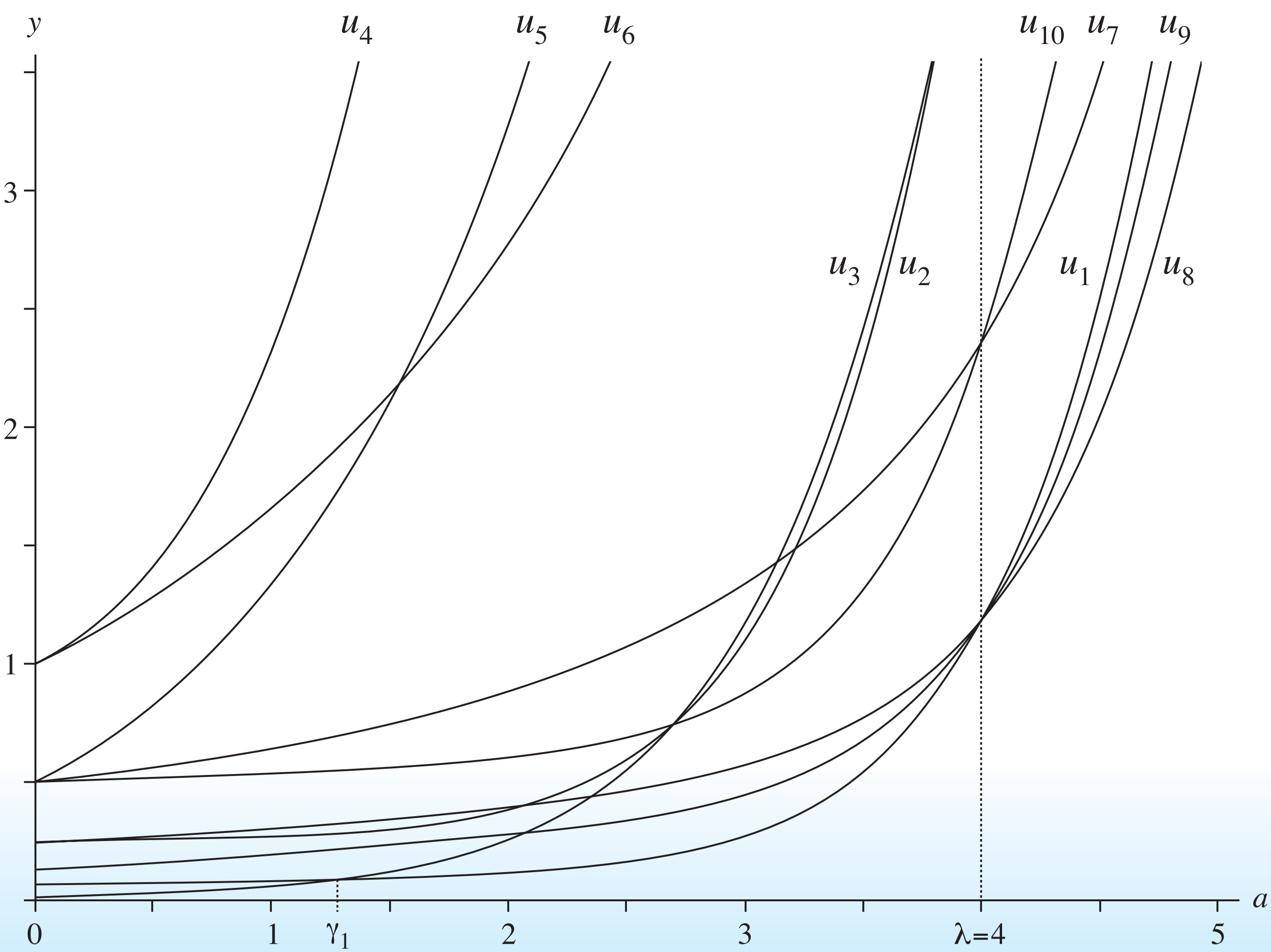
- For any rotation w_j of w_1 and any $a \in (\lambda, \infty)$,

  $x_j = v(a)u_j(a) < v(a)u_1(a) = x_1 < 0$ iff $u_1(a) < u_j(a)$.

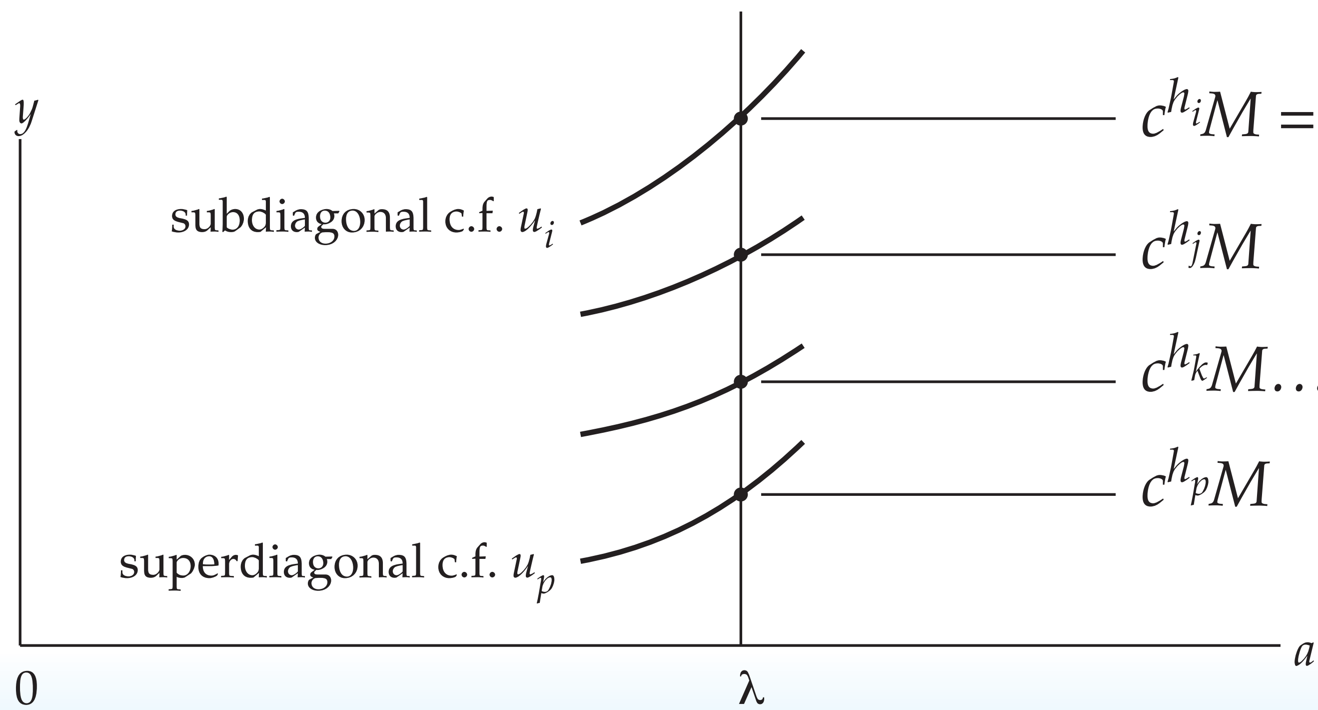
Example of code functions



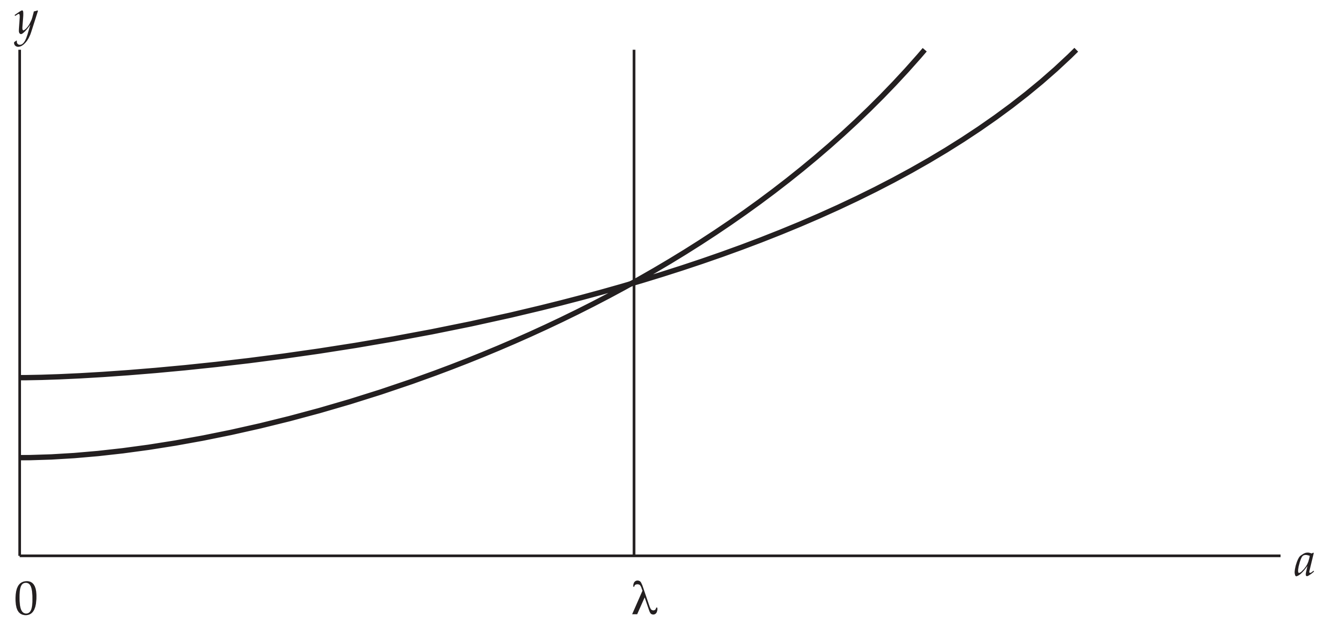
$D_1 = (4, 2, 6, 0, 1, 0, 1, 2, 3, 1), c = 1/2, \alpha = 2.$

Ordering of the code functions at $a = \lambda$

If $0 = h_i < h_j < h_k < \ldots < h_p$ then:

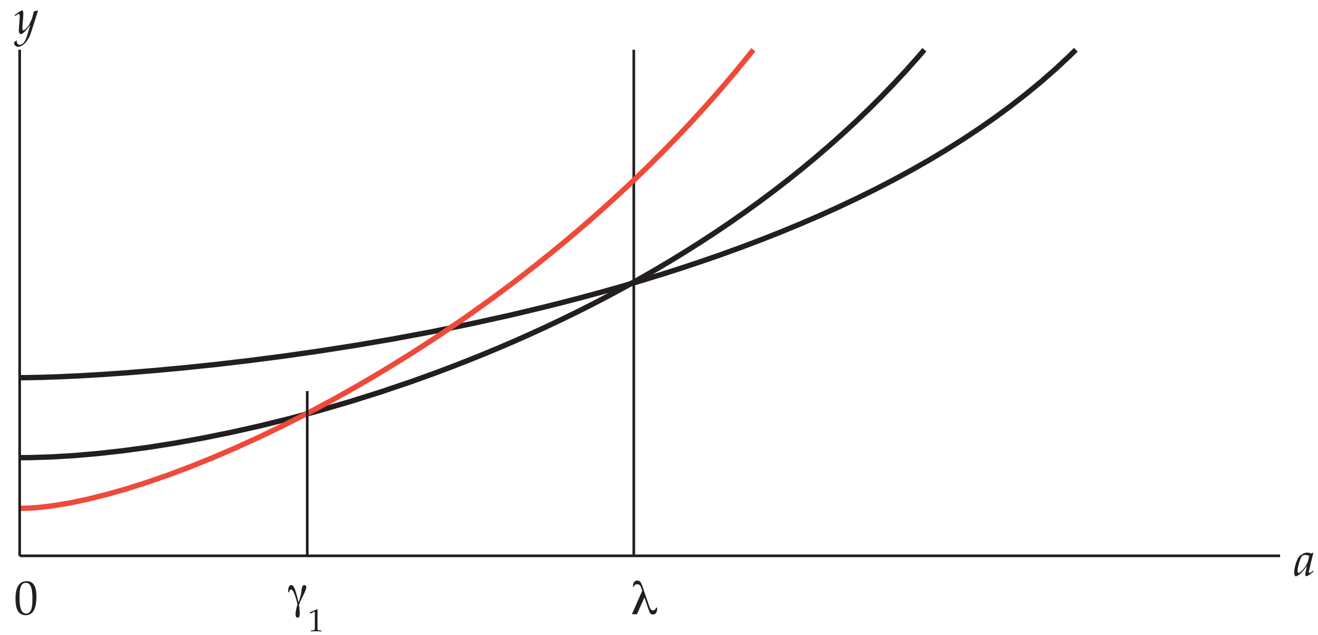


Minimal code functions for a given w



- Superdiagonal code functions around $a = \lambda$.

Minimal code functions



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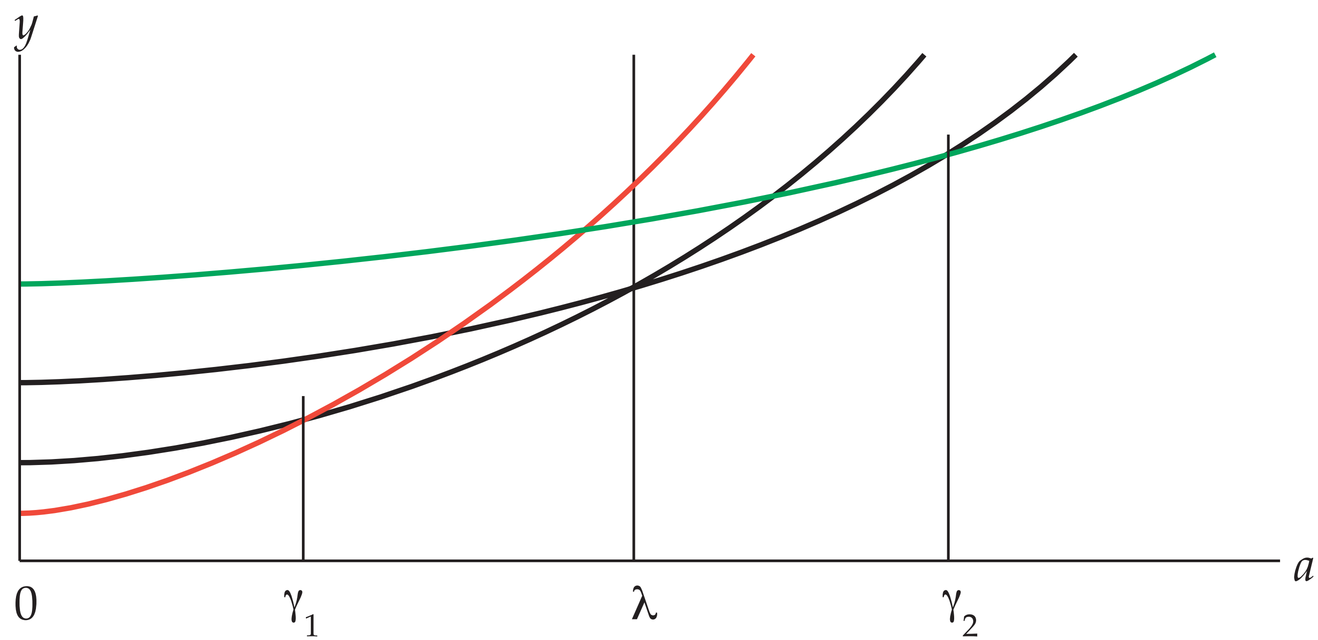
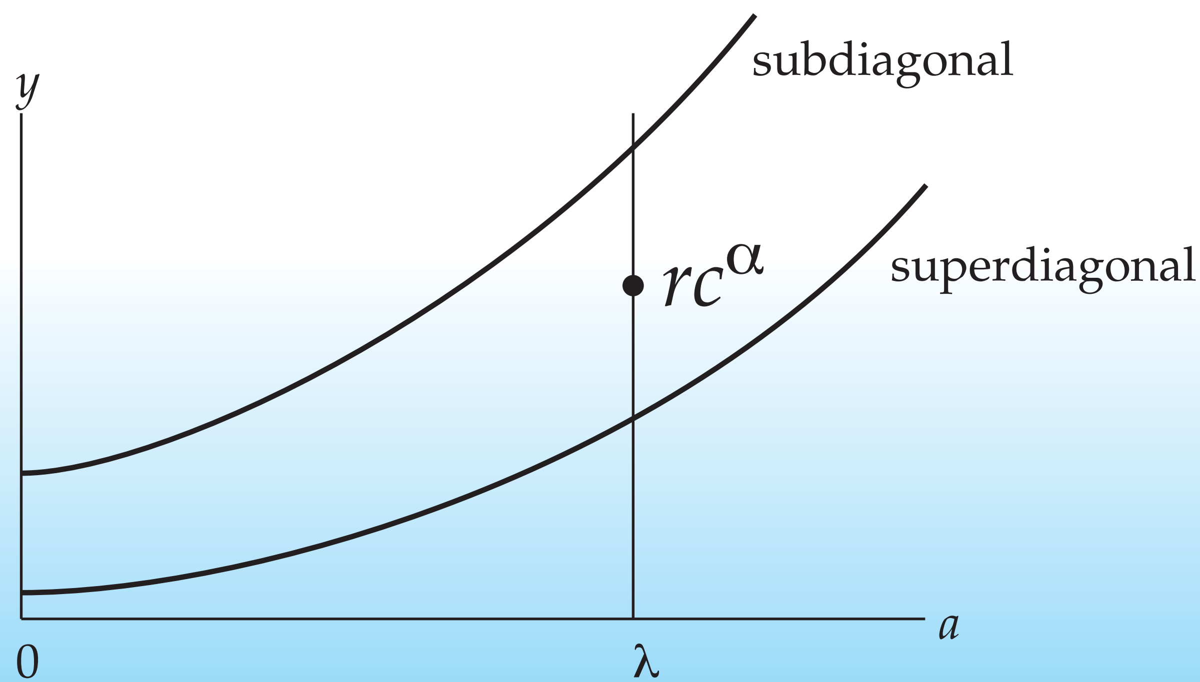


- Superdiagonal code functions around $a = \lambda$.
- Inner minimum function occurs when max d_j is not the first term of the gaps vector D_i (sort of).
- Outer minimum function occurs when min d_j is not the last term of the gaps vector D_i (sort of).

Code function bounds

For all words of a given density α, at $a = \lambda$:

- their superdiagonal code functions are bounded above by rc^{α}, and
- their subdiagonal code functions are bounded below by rc^{α}.



Proving the Maximal Perigee Property

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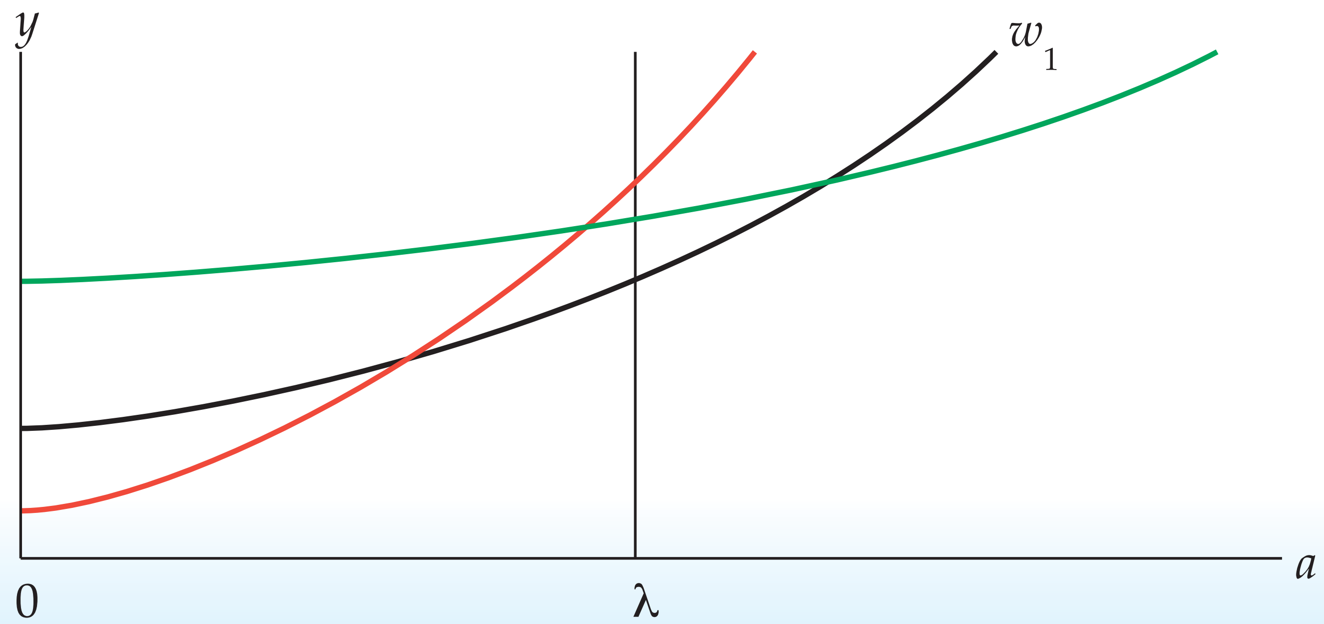
- Thus we must minimize the offset h_1.
- Offset is minimized when
$$Q_1 = (\lceil \alpha \rceil, \lceil 2\alpha \rceil, \lceil 3\alpha \rceil, \ldots, \lceil r\alpha \rceil).$$

Proving the Maximal Perigee Property

- Q_1 is unique, so w_1 is the only superdiagonal word among its rotations.



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- It can be shown that if $Q_1 = (\lceil\alpha\rceil, \lceil 2\alpha\rceil, \lceil 3\alpha\rceil, \ldots, \lceil r\alpha\rceil)$, then in Q_i the jth term is

$$q_{i,j} = \lceil (i+j-1)\alpha \rceil - \lceil (i-1)\alpha \rceil$$

for $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, r$.

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- Since $\lceil x \rceil + \lceil y \rceil \geq \lceil x + y \rceil$ for nonnegative x and y,

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- Fine, so $q_{1,j} \geq q_{i,j}$. But it turns out that if we have equality for every j, then the density of w_1 must be 0.
- Thus $q_{1,j} > q_{i,j}$ for at least one j, so

$$\frac{b}{1-(ac^{\alpha})^r} \sum_{j=1}^{r} c^{q_{1,j}} a^{j-1} < \frac{b}{1-(ac^{\alpha})^r} \sum_{j=1}^{r} c^{q_{i,j}} a^{j-1}$$

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- Hence x_1 is less than any other perigee point among the rotations of w_1. We conclude that w_1 is the maximal perigee word over all $a \in \mathbf{R}^+$.

How to grow perigees: step 0

max peri word	in layer	$T(x)$ max perigee
gf	2,1	1
$gfgf$	4,2	1
$gfgfgf$	6,3	1
$(gf)^k$	$2k,k$	1

In general, $w = (gf)^k$ has perigee $\dfrac{bc}{1-ac}$.

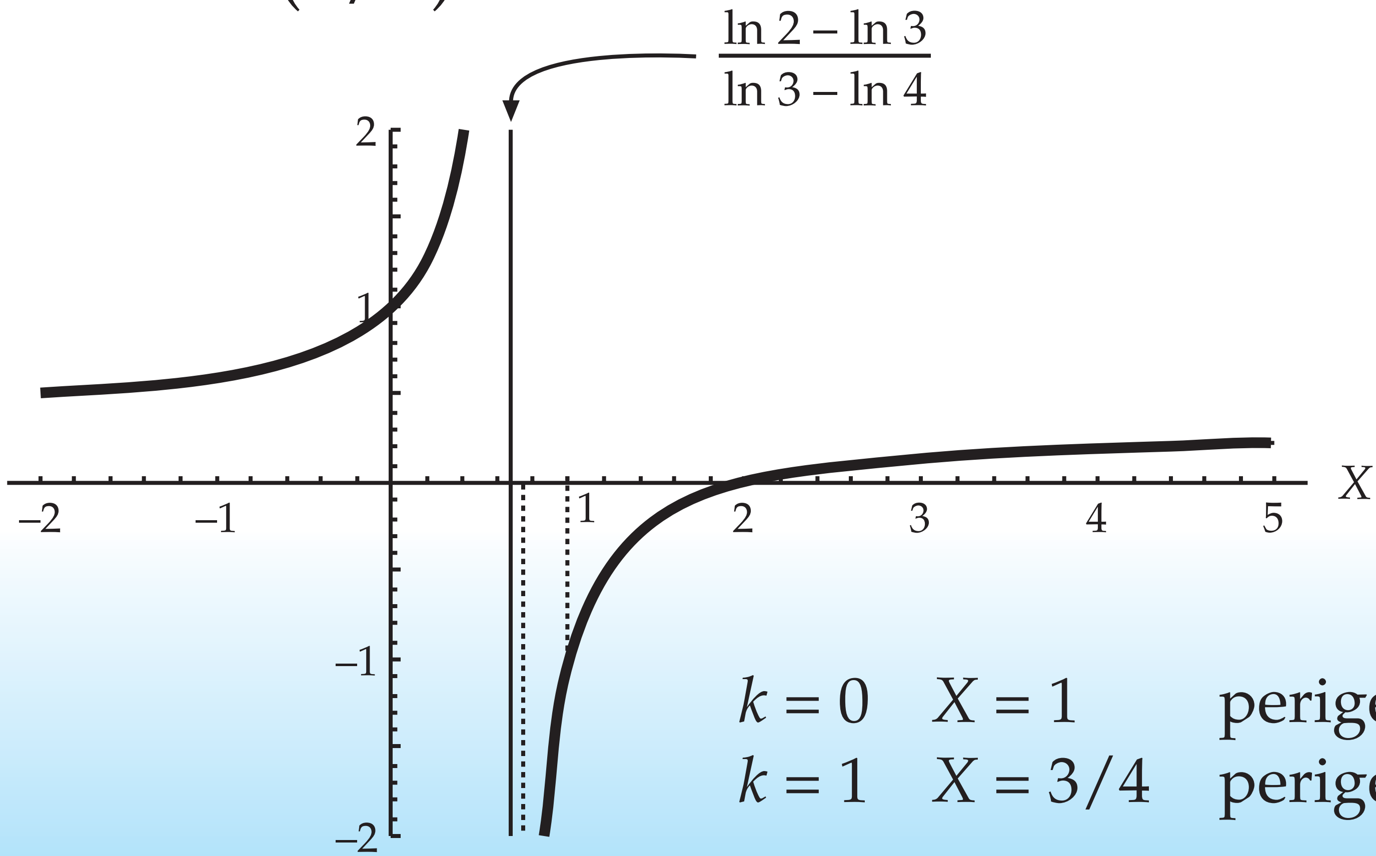
How to grow perigees: step 1

max peri word	in layer	$T(x)$ max perigee
f	1,1	–1
$(gf)f$	3,2	–5
$(gf)^2f$	5,3	4.6
$(gf)^3f$	7,4	2.15
$(gf)^4f$	9,5	1.6
$(gf)^kf$	$2k+1\,,\,k+1$	

Perigee family: step 1

In the $T(x)$ case, $w = (gf)^k f$ has perigee $\dfrac{2 - X}{2 - 3X}$

where $X = (3/4)^k$



$k = 0 \quad X = 1 \quad$ perigee = –1

$k = 1 \quad X = 3/4 \quad$ perigee = –5

How to grow perigees: step 4

max peri word	in layer	$T(x)$ max perigee
$ffff$	4,4	–1
$(gf)ffff$	6,5	–1.178
$(gf)ff(gf)ff$	8,6	–1.727
$(gf)f(gf)f(gf)ff$	10,7	–2.493
$(gf)f(gf)f(gf)f(gf)f$	12,8	–5
$(gf)^2f(gf)f(gf)f(gf)f$	14,9	–12.449
$(gf)^2f(gf)f(gf)^2f(gf)f$	16,10	24.5

How to grow perigees: step 4

- Insert (gf) to the left of $f_{1+\lfloor \frac{4}{1} \cdot 0 \rfloor}$

  Result: (gf)$ffff$.

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  Result: $(gf)ff(gf)ff$.

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- Insert (gf) to the left of $f_{1+\lfloor \frac{4}{3} \cdot 0 \rfloor}, f_{1+\lfloor \frac{4}{3} \cdot 1 \rfloor}, f_{1+\lfloor \frac{4}{3} \cdot 2 \rfloor}$

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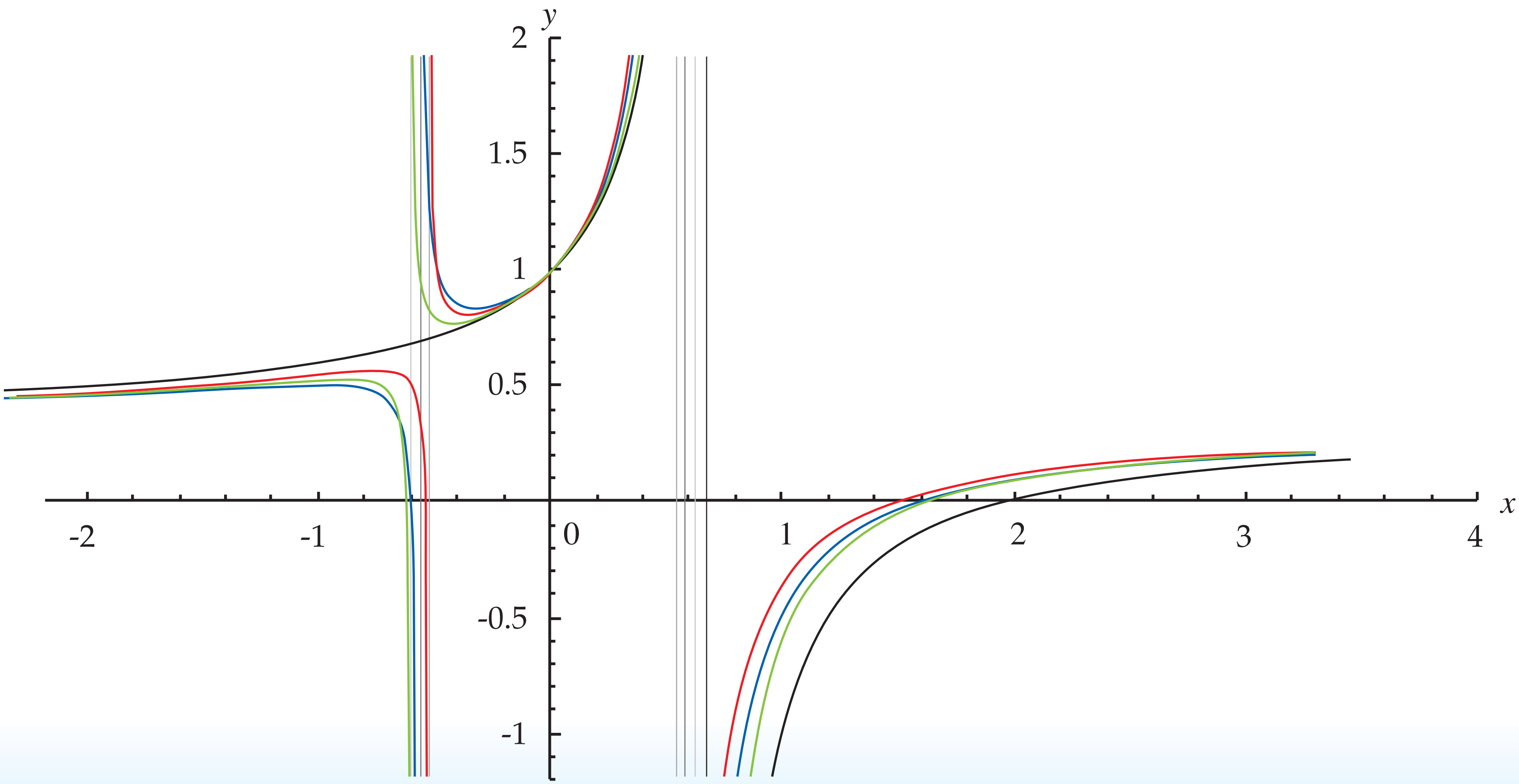
  Result: $(gf)f(gf)f(gf)f(gf)f$.

Perigee families: step 4

In the $T(x)$ case, where again $X = (3/4)^k$:

perigee words of this form		*lie on this curve*
$w = (gf)^k f (gf)^k f (gf)^k f (gf)^k f$	$\rightarrow$	$\dfrac{2 - X}{2 - 3X}$
$w = (gf)^{k+1} f (gf)^k f (gf)^k f (gf)^k f$	$\rightarrow$	$\dfrac{1 + X + 2X^2 + 4X^3 - 4X^4}{1 - 12X^4}$
$w = (gf)^{k+1} f (gf)^k f (gf)^{k+1} f (gf)^k f$	$\rightarrow$	$\dfrac{1 + X - X^2}{1 - 3X^2}$
$w = (gf)^{k+1} f (gf)^{k+1} f (gf)^{k+1} f (gf)^k f$	$\rightarrow$	$\dfrac{4 + 4X + 6X^2 + 9X^3 - 9X^4}{4 - 27X^4}$

Perigee families: step 4



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- Can we use the rational functions of a perigee family to find integer perigees, or prove their nonexistence?
- Does all of this just recast the $3x + 1$ problem in yet another inscrutably impenetrable form?

"I am inclined to believe that at the root of all deep mathematics there is a combinatorial insight."

—P. R. Halmos

Fifty years of linear algebra: a personal reminiscence. In *Visiting Scholars' Lectures—1987*, Texas Tech University Mathematics Series **(15)**, 71–89.