MODEST TRIANGLES AND PERIAMBIC POINTS

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Abstract. The major periambic points P and Q are radical centers associated with the six circles having a triangle's vertices as centers and its sides as radii. We offer an alternate construction of P and Q which is valid for "modest" triangles, in which the ratio of two side lengths lies in the closed interval $[-1+\sqrt{2}, 1+\sqrt{2}]$.

The radical center of three circles is the point of concurrency of the three radical axes determined by pairs of the circles. The major periambic points P and Q are radical centers associated with a triangle ABC (labeled counterclockwise, with sides $a = BC$, $b = CA$, and $c = AB$): P is the radical center of the periambic circles P_A, P_B, and P_C, having centers A, B, and C and radii $AB = c$, $BC = a$, and $CA = b$, respectively; P lies on each of the radical axes p_a, p_b, and p_c (called p-lines) determined by circle pairs $\{P_B, P_C\}$, $\{P_C, P_A\}$, and $\{P_A, P_B\}$, respectively. Q is the radical center of periambic circles Q_A, Q_B, and Q_C, with centers A, B, and C, and radii AC, BA, and CB, respectively; Q lies on each of the radical axes q_a, q_b, and q_c (called q-lines) determined by circle pairs $\{Q_C, Q_B\}$, $\{Q_A, Q_C\}$, and $\{Q_B, Q_A\}$, respectively. These definitions are given in [2], where it is shown, for instance, that P, Q, the orthocenter, and the circumcenter of triangle ABC form a parallelogram.

In [2], "perpendicularity" is a dominant theme; each of the 12 real radical axes defined by pairs of periambic circles is at right angles to a side of triangle ABC. The configuration shown in Figure 1, and summarized in the following conjecture due to Marty Getz [1], offers a step toward defining the points P and Q by means of non-perpendicular elements. We bear in mind that the cyclic relabeling $(a, A) \to (b, B) \to (c, C) \to (a, A)$ of sides and vertices yield in total three versions of each result below; for brevity and clarity, only one version will be presented.

Conjecture. *Given triangle ABC, let p_a intersect P_A at J_A (on the same side of BC as A) and at K_A. Let CJ_A and CK_A intersect P_B at L_B and M_B, respectively. Then L_B, P, and M_B are collinear.*

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This conjecture could be a theorem, except for one problem: its first sentence. As Figure 2 shows, the radical axis p_a may not intersect the circle P_A.

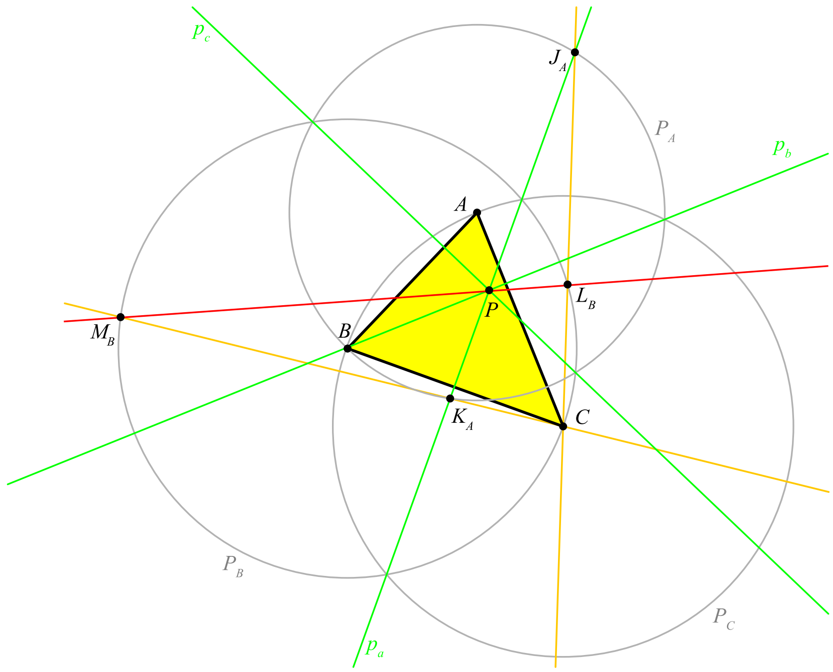

FIGURE 1. Radical axis p_a intersects periambic circle P_A at J_A and K_A. CJ_A and CK_A intersect P_B at L_B and M_B, respectively. The conjecture claims that L_B, P, and M_B are collinear.

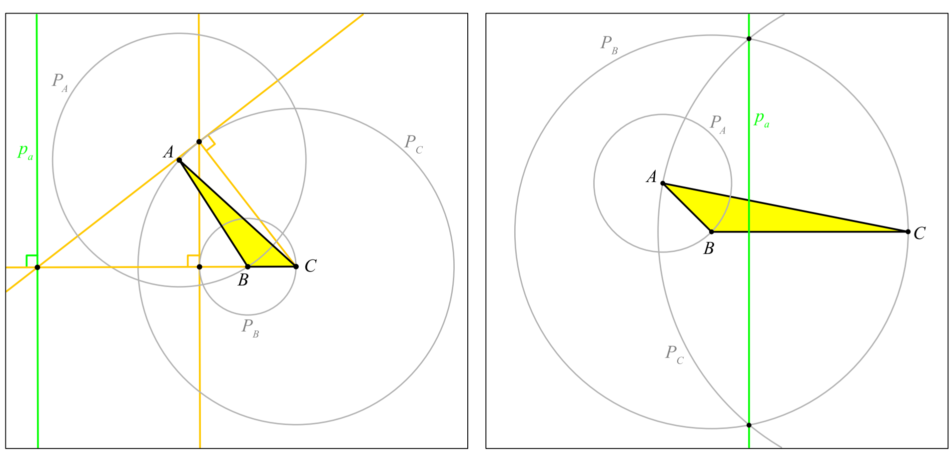

FIGURE 2. Two cases in which p_a and P_A fail to intersect. Gold lines on the left are for locating p_a.

Modest triangles

To work the conjecture into something provable, we pause to investigate the conditions under which the points J_A and K_A are well-defined.

Proposition 1. *In triangle ABC, the radical axis p_a intersects the periambic circle P_A if, and only if,*

$$-1+\sqrt{2} \leq \frac{c}{a} \leq 1+\sqrt{2} \tag{1}$$

or, equivalently,

$$-1+\sqrt{2} \leq \frac{a}{c} \leq 1+\sqrt{2}\,. \tag{2}$$

We call a triangle *ca-modest* if it satisfies inequality (1). By condition (2), a ca-modest triangle is also ac-modest. Apropos of our comment on cyclic relabeling of vertices and sides, we note that, in an ab-modest triangle, p_b and P_B intersect; in a bc-modest triangle, p_c and P_C intersect. A triangle in which all three p-lines intersect their corresponding periambic circles, as in Figure 1, will simply be called modest.

Proof. Conjugate multiplication in the reciprocal of (1) yields (2); we prove the latter. Without loss of generality, in the cartesian plane let A, B, and C have coordinates $(x_1, x_2), (0,0)$, and $(a,0)$, respectively, with $x_2 > 0$ and $a > 0$. Let p_a intersect the x-axis at $T:(t,0)$. Since T has equal powers with respect to P_B and P_C, we have

$$\begin{aligned} BT^2 - a^2 &= CT^2 - b^2 \\ t^2 - a^2 &= (a-t)^2 - b^2 \\ t &= \frac{2a^2 - b^2}{2a}\,. \end{aligned} \tag{3}$$

The coordinates of A, considered as a point on P_C, satisfy $(a-x_1)^2 + x_2^2 = b^2$. Substituting for b^2 in (3) and rearranging, we obtain

$$\begin{aligned} t &= \frac{2a^2 - \left((a-x_1)^2 + x_2^2\right)}{2a} \\ &= \frac{a^2 + 2ax_1 - (x_1^2 + x_2^2)}{2a} \\ &= \frac{a^2 + 2ax_1 - c^2}{2a}\,. \end{aligned} \tag{4}$$

Let $U:(u, x_2)$ and $V:(v, x_2)$ be the left and right endpoints, respectively, of the diameter of P_A which is parallel to the x-axis. Clearly

$$u = x_1 - c \quad \text{and} \quad v = x_1 + c\,.$$

For p_a to intersect P_A, we require

$$u = x_1 - c \le t \le x_1 + c = v\ . \tag{5}$$

The left-hand inequality becomes

$$\begin{aligned} x_1 - c &\le \frac{a^2 + 2ax_1 - c^2}{2a} \\ 0 &\le a^2 + 2ac - c^2\ , \end{aligned} \tag{6}$$

which, for nonzero c, yields

$$a \ge c(-1 + \sqrt{2})\ . \tag{7}$$

Similarly, the right-hand inequality in (5) yields

$$\begin{aligned} \frac{a^2 + 2ax_1 - c^2}{2a} &\le x_1 + c \\ 0 &\le -a^2 + 2ac + c^2\ , \end{aligned} \tag{8}$$

having positive solution

$$a \le c(1 + \sqrt{2})\ . \tag{9}$$

Results (7) and (9) combine to establish inequality (2). Conversely, assuming (1), we have

$$\begin{aligned} a^2(3 - 2\sqrt{2}) &\le c^2 \text{ and} \\ 2a^2(-1 + \sqrt{2}) &\le 2ac\ , \end{aligned}$$

from which we recover (8); likewise, assuming (2), we can construct (6). It then follows that (5) is satisfied. □

Figure 3 shows the configurations resulting when c/a attains the values $-1+\sqrt{2}$ and $1 + \sqrt{2}$. In this construction,

$$c = AB = BG = GH = HC\ ,$$

and $\angle GHC$ is a right angle, so $a = BC = c + c\sqrt{2}$; the radical axis p_a (solid green line) is tangent to P_A on the circle's right. On the other hand, in triangle ABC' where a', b', c' are the sides opposite A, B, C', respectively, C' is constructed so that $a' = BC' = -c + c\sqrt{2}$, and the resulting radical axis p'_a (dotted green line) is tangent to P_A on its left. From the ratios c/a and c'/a' and the shared angle CBA, it is clear that triangles ABC and ABC' are inversely similar.

One can construct an isosceles triangle in which two of the three p-lines are tangent to their corresponding periambic circles. In Figure 4,

$$b = c = a(1 + \sqrt{2})\ ,$$

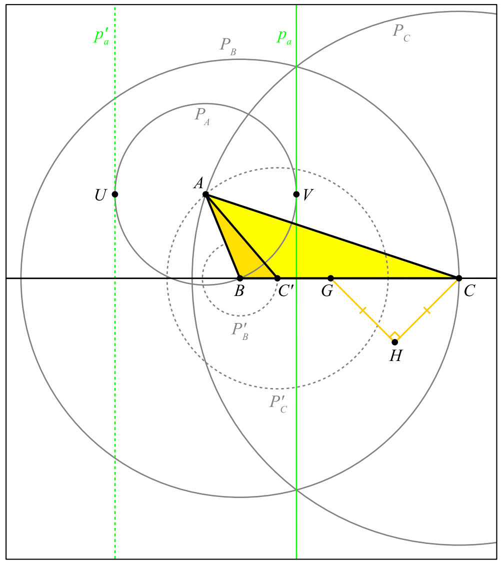

FIGURE 3. Example showing ca- and $c'a'$-modest triangles ABC and ABC', respectively, for which the extrema of the ratios of Proposition 1 are achieved when A, B, and $\angle CBA = \angle C'BA$ are fixed. Solid green line p_a is the radical axis of solid gray circles P_B and P_C; dotted green line p'_a is the radical axis of dotted gray circles P'_B and P'_C.

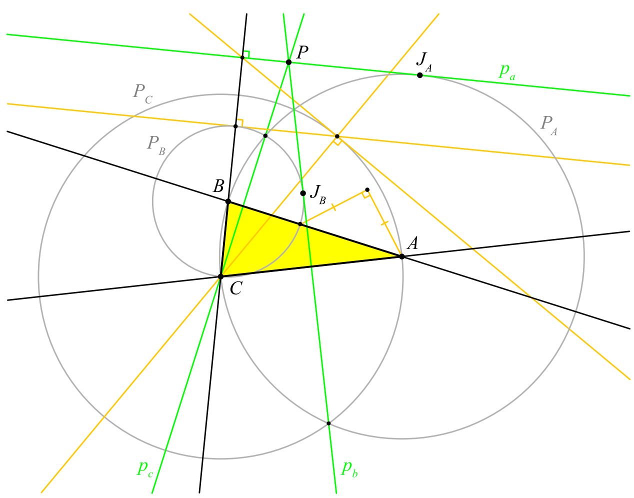

FIGURE 4. Isosceles triangle ABC in which $b = c = a(1 + \sqrt{2})$. In this case, p_a and p_b are tangent to P_A and P_B, respectively. For any shorter side a, p_a and p_b do not intersect their corresponding periambic circles.

and Proposition 1 is satisfied by a/b, b/a, c/a, and a/c. If B were nearer C than is shown in Figure 4, p_a would not intersect P_A, nor would p_b intersect P_B.

A theorem from the limited conjecture

Restricted to the relevant type of modest triangle, the conjecture which began this note becomes a theorem.

Theorem 1. *Let ABC be a ca-modest triangle. Let p_a intersect P_A at J_A (on the same side of BC as A) and at K_A. Let CJ_A and CK_A intersect P_B at L_B and M_B, respectively. Then L_B, P, and M_B are collinear.*

Proof. By requiring triangle ABC to be *ca*-modest, we guarantee the existence of the points J_A and K_A, per Proposition 1. Figures 5 and 6 illustrate the following construction for the cases in which P_B and P_C do and do not intersect, respectively. Let L_BM_B intersect radical axis p_a at Z. By definition, Z has equal powers with respect to circles P_B and P_C. Let S, different from C, be the intersection point of

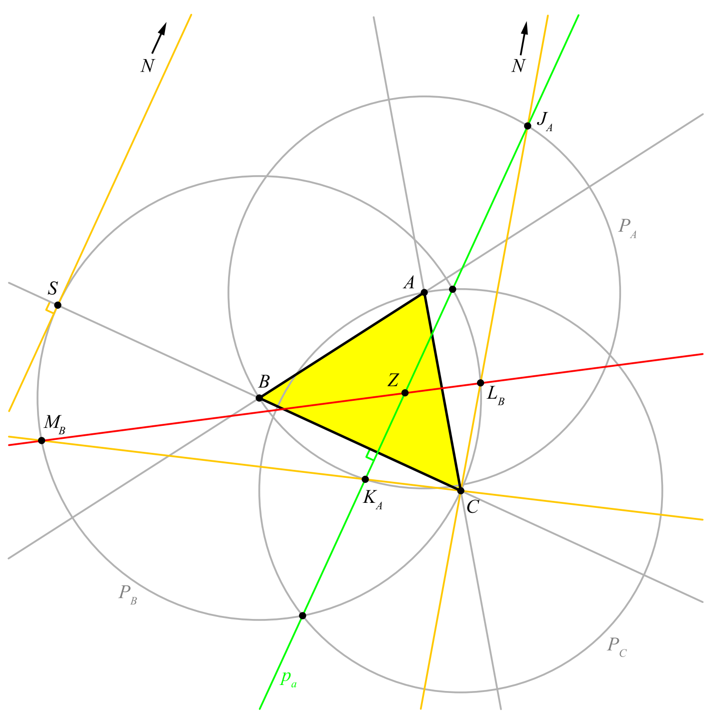

Figure 5. Construction for the proof of Theorem 1, in which P_B and P_C intersect.

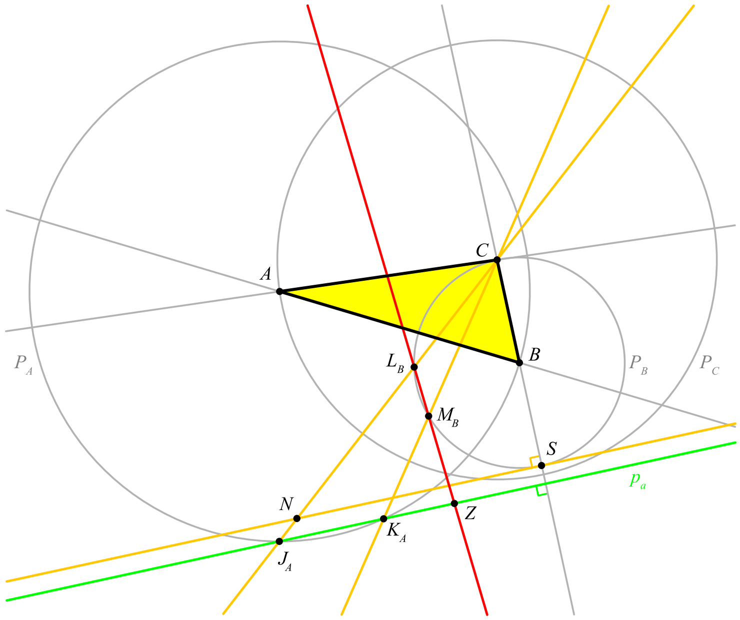

FIGURE 6. Construction for the proof of Theorem 1, in which P_B and P_C do not intersect.

BC with P_B, and let the tangent to P_B at S intersect CJ_A at N. Arcs and angles measured counterclockwise are positive, so we have

$$\begin{aligned}\angle ZJ_AL_B = \angle SNC &= \frac{1}{2}\left(\operatorname{arc} SC - \operatorname{arc} L_BS\right)\\ &= \frac{1}{2}\operatorname{arc} CL_B = \angle CM_BL_B\\ &= \angle K_AM_BZ\ .\end{aligned}$$

From this we see that $\triangle ZM_BK_A$ and $\triangle ZJ_AL_B$ are inversely similar. Therefore,

$$\frac{ZK_A}{ZM_B} = \frac{ZL_B}{ZJ_A}$$

or

$$ZM_B \cdot ZL_B = ZK_A \cdot ZJ_A\ . \tag{10}$$

The power ρ of point Z with respect to circle P_B is

$$\rho = ZM_B \cdot ZL_B\ , \tag{11}$$

so from Equations (10) and (11) we have

$$\rho = ZM_B \cdot ZL_B = ZK_A \cdot ZJ_A\ .$$

Since $ZK_A \cdot ZJ_A$ is the power of Z with respect to circle P_A, we conclude that Z lies on the radical axis of P_A and P_B, and hence is the radical center of circles P_A, P_B, and P_C, i.e., Z is the major periambic point P. □

By remarks following Proposition 5 in [2] (concerning the radial symmetry of the major periambic points P and Q around the midpoint of the segment joining the orthocenter and circumcenter), there is a corresponding result for Q. Identical to Theorem 1, but using Q instead of P and reversing the roles of B and C, we state it here for completeness.

Theorem 2. *Let q_a intersect Q_A at J_A (on the same side of BC as A) and at K_A. Let BJ_A and BK_A intersect Q_C at L_C and M_C, respectively. Then L_C, Q, and M_C are collinear.*

References

[1] M. Getz, personal communication, March 2003.

[2] D. J. Jones, The periambic constellation: altitudes, perpendicular bisectors, and other radical axes in a triangle. *Forum Geometricorum* **17** (2017) 383–399.

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